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ON THE CONDITIONAL VARIANCE - COVARIANCE FOR SCALE MIXTURES OF NORMAL DISTRIBUTIONS AND ITS APPLICATIONS

Abstract

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Let $\mathbf{X} = A^{1/2}\mathbf{G}$ be a scale mixture of multivariate normal distribution with $\mathbf{X}$, $\mathbf{G} \in \mathbf{R}^n$, where $\mathbf{G}$ is a multivariate normal vector, and A is a positive random variable independent of the multivariate random vector $\mathbf{G}$. This model is capable of capturing the frequently reported leptohurtosis in economic data. This thesis focused on the investigation of the conditional variance-covariance of the scale mixtures model under some regularity conditions. We found that the conditional variance-covariance, $Cov(\mathbf{X}_2|\mathbf{X}_1)$, $\mathbf{X}_1 \in \mathbf{R}^m$, is always finite *a.s.* for m greater or equal to 2, where $\mathbf{X}_1$ is m-dimensional vector and $m<n$. It remains finite *a.s.* for m=1, if and only if $E\left[A^{1/2}\right] < \infty$. It was shown that the conditional variance is not degenerate as in the Gaussian case, instead, it is a function of expectation of mixing variable A conditioning on $\mathbf{X}_1$. This function, denoted, by $S_{A,m}(.)$, depends upon $\mathbf{x}_1$, the mixing variable A, and the dimensionality m as well. In this study, integral representation forms of $S_{A,m}(.)$ were pre-

sented, and various properties were derived based on the integral representations. Applications to uniform mixture, $\alpha/2$-stable mixture and generalized gamma mixture of normal distributions were also given. Some asymptotic expansions with error bounds for $S_{A,m}(.)$ were obtained using Laguerre and Hermite pplynomials. All these asymptotic expansions were presented in manageable and computable forms. The results provided in this research will help us to better understand the behaviors of the heteroskedasticity in regression when the errors assume the structure of normal scale mixture.

In this thesis, we also developed the asymptotic theory of sample moments and some unit root statistics, such as, the Lagrange multiplier statistics, the Durbin-Watson statistics, and the ranked Dickey-Fuller statistics, for the first-order autoregressive process with the innovations belonging to the domain of attraction of symmetric stable law. We established the limiting theories in terms of standard $S\alpha S$ *Levy* motions. Spurious regression phenomenon was also investigated in the context of infinite variance. These asymptotic results can be viewed as parallel extensions of the Gaussian case, and may be applied in the investigation of integration or cointegration for the heavy-tailed time series.

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CHAPTER 1

INTRODUCTION

1.1 Motivations of the Study

Mixture distributions have been proved to be of considerable interest in recent years in terms of both methodological development and applications. Researchers have found far-reaching applications ranging from finance to economics, from physics to biology, and from decision theory to reliability theory. Mixture model is attractive when a distribution under investigation is too complicated to work with, but can be decomposed as mixture of simple (known) distributions. Among a large variety of mixture distributions, the family of scale (variance) mixtures of normal distributions is of particular interest because it is closely related to normal theory. A wide class of continuous, symmetric, unimodal distributions on the real line can be expressed as a scale mixture of normal distributions. Examples include the Student's t family, Laplace's double exponential, logistic, the exponential power family, the α-stable family, and the contaminated normal family, etc.. All these important distributions share one common feature: they have heavier tails relative to normal distribution, and are often viewed as good candidates in modeling economic data which exhibits leptokurtosis. A better understanding of this model will be helpful in modeling heavier tailed data. In the linear regression model, if the errors are not normal, then constant variance assumption is often violated even though the linear regression property may still holds. In this case, we encounter the heteroscedasticity. As shown in Chapter 1, the scale mixture of normal distributions provides a good example for heteroscedastictiy in regression. A thorough study of this mixture model will help us understand the heteroscedasticity phenomenon in a regression model when the errors are assumed to have the scale mixture of normal distributions.

Another motivation for this study arises from financial modeling. There are numerous empirical evidence against the normality assumption for the marginal distribution of stock returns and price changes in common stocks and foreign exchanges. However, the stationarity of stock returns remains a crucial assumption in estimating expected returns under the Capital Asset Pricing Models (*CAPM*) as well as in option pricing models. Accordingly, the uncorrected heteroscedasticity will result in biased estimators of variance, and such biased estimators are likely to lead to inferences which are misleading at best. Thus, the detection of the sources of heteroscedaticity in common stock returns and price changes will be helpful in explaining variance of the stock returns. Since the seminal work of Mandelbrot (1963a, 1963b), how to model the observed leptokurtosis and heteroschedasticity has been a popular topic in empirical financial studies. To explain the observed leptokurtosis and heteroscedasticity, many mixture models have been proposed and tested. For example, Mandelbrot (1963a, 1963b, 1967) and Fama (1965) suggested the use of α-stable distributions; Blattberg and Gonedes (1974) claimed Student t distribution has a better fit than normal and stable distributions; Clark (1974) used the subordinated stochastic process to model the stock return generating process; Ball and Torous (1985), Akriray and Booth (1986, 1987) advocated a Poisson jump-diffusion process; Kon (1984) proposed a finite (discrete) mixture of normal distribution for stock returns; Gray and French(1990) used the exponential power family to model stock returns; Smith (1981) tested the hypothesis of logistic distribution for stock returns, etc.. All these distributions are successful in capturing the observed leptokurtosis more or less. One remaining question is that these distributions proposed and tested by the above authors are *ad hoc* distributions. There is no theoretical justifications for the use of the above mentioned distributions. Therefore an intuitive model, which is able to explain the empirical features and can provide some natural explanation for the possible generating process of leptokutosis and heteroscedasticity, is in need. Phillips (1995) argued that leptokurtosis may come either from random summation of *iid* normal variables or from randomization of scale parameter in normal distribution. These two schemes result in the normal scale mixtures. If the underlying data generating process is Gaussian, but "contaminated" (or, mixed) by some other unknown process, the resulting process will exhibit non-homogeneity, and the marginal distribution will be leptokurtotic. Scale mixture of normal hypothesis provides a natural way to explain how the data generating process is contaminated. The mixture of normal distributions hypothesis is thus both theoretically and empirically appealing to financial research community.

Epps and Epps (1976) and Akgiray (1989) argued that the heteroscedasticity in common stock returns is a function of the information arrival to the market. Since 1980's, scale mixture of normal distributions has received an increasing amount of attention in modeling volatility of stock returns and price changes. For example, the *ARCH-GARCH* family:

$$r_t = r_0 + \varepsilon_t, \tag{1.1.1}$$

$$\varepsilon_t = \sigma_t Z_t, \tag{1.1.2}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i} + \sum_{j=1}^{p} \gamma_j \sigma_{t-j}^2 \tag{1.1.3}$$

uses normal scale mixture to capture the conditional heteroscedasticity (equation 1.1.2), and uses the autoregressive process to model volatility persistence and clustering (equation 1.1.3). Several recent studies (Tauchen and Pitts 1983, Harris 1982, 1986, and 1987, Foster and Viswanathan 1990, 1993, Richardson and Smith 1994) focused on the empirical tests of the mixture of distributions hypothesis in which the stochastic mixing variable has certain known distributions such as inverted gamma, log-normal, and symmetric α-stable. Strong empirical evidence in favor of mixture of distributions hypothesis for daily stock price changes was found in those studies. Another popular topic in empirical finance is the investigation of the relationship between prices variability (measured as squared price changes) and trading volume under the following bivariate mixture model:

$$\begin{aligned} \Delta P_t &= \sqrt{I_t}\,\sigma_1 Z_t, \\ Vol_t &= cI_t + \sqrt{I_t}\,\sigma_2 Z_t, \\ Cov\left(\Delta P_t,\ Vol_t \middle| I_t\right) &= 0, \end{aligned} \tag{1.1.4}$$

where I_t is a positive random variable denoting the relevant information flow arriving at stock market, Z_t is a Gaussian process independent of information flow. In the study of the relationship between ΔP_t^2 and Vol_t, one needs to evaluate the conditional variance $E\left[\Delta P_t^2 \middle| Vol_t\right]$, which, in turn, depends on the functional form of $E\left[I_t \middle| Vol_t\right]$. Note that the marginal distribution of information flow is in general unknown, it is impossible to evaluate $E\left[I_t \middle| Vol_t\right]$ explicitly without assuming *a prior* distributional form for the information flow I_t. Harris (1987) used the number of transactions as the proxy for the information flow. Such proxy was found imperfect in Richardson and Smith (1994). How to evaluate $E\left[I_t \middle| Vol_t\right]$, even approximately, remains challenge to financial econometric researchers, and motivates us to investigate the behavior of conditional variance for the scale mixture of normal distributions.

1.2 Purposes of the Study

There is a large body of literature on the scale mixture of normal distributions, but most of them are confined to the discrete mixture or the univariate case. In this study, we want to investigate the properties of normal mixture model in a multivariate setup. In a multivariate linear regression model, as shown in Chapter 1, if the errors have scale mixture of multivariate normal distribution, the regression property holds, that is, the conditional expectation remains linear. But the conditional variance of **Y** given **X** is no longer degenerate, and the non-degenerate conditional variance accounts for the heteroscedasticity in the regression model. One of major interests in this study is thus placed on the investigation of the behaviors of conditional variance under the structure of normal scale mixture based on the integral representations of the conditional variance.

The second purpose of this study is to find the asymptotic forms of conditional variance for some important mixing schemes, such as α-stable mixture, uniform mixture, and Gamma mixture, around both small argument and large argument.

Since the distribution of mixing variable is unknown in general, it is impossible to evaluate the conditional variance exactly. However, without assuming distributional from of mixing variable, we may still be able to evaluate the conditional variance approximately based on the integral representations. Our third purpose in this study is to evaluate the conditional variance under the structure of multivariate normal mixtures approximately using some special functions.

The fourth purpose of this study is to investigate the asymptotic behaviors of sample moments and unit root test statistics for testing H_0: $\rho = 1$ in the following first-order autoregressive model with heavy tails:

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \qquad (1.2.1)$$

where ε_t is a sequence of *iid* random variables from the domain of attraction of a symmetric stable law. Many time series data in finance and economics exhibits non-stationarity due to a unit root. How to detect the presence of unit roots has been a hot topic in econometric literature. For the scale mixture of normal innovations, i.e., $\{\varepsilon_t, t \geq 0\} =_d \{A^{1/2} G_t, t \geq 0\}$, we find that if they have finite variance, then the asymptotic results would be the same as they are for the Gaussian case. If ε_t's have infinite variance, that is, ε_t 's are sub-Gaussian, the asymptotic distributions of scale invariant statistics such as t or F, and most of unit root test statistics as well will be the same as they are for the Gaussian case, since they are radically decomposable (Ng and Fraser 1994). Thus there is no need to further study the asymptotic theory of unit root test statistics for the scale mixture of normal errors. Our study hence is confined to the case that the innovations are independent and identically distributed random variables belonging to the domain of attraction of a $S\alpha S$ law, so they have infinite variance but can not be sub-Gaussian.

1.3 Outline of the Thesis

This thesis is a collection of several papers and can be divided into two parts. The first part (Chapter 2 - 5) consider the properties of the scale mixture of normal distributions, and the behaviors of condition variance under the multivariate normal scale mixture structure. The second part (Chapter 6) deals with the asymptotic theory of unit root test for the autoregressive model with infinite variance.

We start with reviewing the relevant literature. Model definitions and assumptions are given in Chapter 2. Some properties are also collected there. In the third chapter, we study the behaviors of conditional variance under the normal scale mixture model. Integral representations are obtained in this chapter.

The asymptotic behaviors of stable mixture, uniform mixture, and Gamma mixture of normal distributions are studied in Chapter 4. In Chapter 5, expansions and error bounds for the conditional variance are established using Laguerre and Hermite polynomials.

In Chapter 6, we investigate the limiting theory of sample moments for a first-order autoregressive process with the innovations belonging to the domain of attraction of a symmetric α-stable law.

Asymptotic distributions for some unit root test statistics such as the Lagrange multiplier statistics, Durbin-Watson statistics, Ranked Dickey-Fuller statistics are provided in this chapter, and the spurious regression in the context of infinite variance is also analyzed in this chapter.

CHAPTER 2

REVIEW OF THE SCALE MIXTURE OF NORMAL DISTRIBUTIONS

A large body of literature exists on scale mixture of normal distributions. Kelker (1971) and Andrews and Mallows (1974) gave necessary and sufficient conditions for a distribution to be a normal scale mixture. This family can be characterized by the property of complete monitonicity. Teicher (1963) addressed the general question of identifiability. Keilson and Steutel (1974) gave the measure of departure from normality in terms of L_1-norm. Basu (1996) showed the class $\mathcal{F}(p, Q; \sigma^2) = \{F \text{ is a normal scale mixture and } F(Q) = p,\ Var_F(X) = \sigma^2\}$ is non-empty if and only if p is greater than or equal to some fixed constant. Review of literature on the scale mixture of univariate distributions can be found in Gupta and Huang (1981), Everitt and Hand (1981), and Titterington (1990). For the scale mixture of multivariate normal distributions, Shimizu (1987, 1995) and Fujikoshi and Shimizu (1989a, 1989b) obtained some asymptotic expansions around standard normal distribution for some mixture of multivariate normals in terms of Hermite polynomials. The error bounds were also evaluated in the L_1-norm in their papers. Huang and Cambanis (1979), Cambanis, Huang and Simons (1981), Hardin, Samorodnisky and Taqqu (1991), Rosinski (1992), Wu and Cambanis (1991), Ng and Fraser(1994), Cambanis and Fotopoulos (1995) approached to this problem in the framework of spherically symmetric distributions.

2.1 The Models

Let $G(a)$ be a *cdf* with the support on Ω, and let $F(x,a)$ be a distribution function in x for each a in the support of G. Assume $F(x,a)$ is Borel measurable in a for every x. Then

$$H_G(x) = \int_{\Omega} F(x,a)\, dG(a) \tag{2.1.1}$$

is a distribution function, called G-mixture of F distribution, and G is referring to as a mixing distribution. When G has a finite support, $H_G(x)$ in (2.1.1) (replacing integration by summation) is called finite G-mixture of F-distribution

We are interested in a special case when a in (2.1.1) is a scale parameter of F

$$H_G(x) = \int_{\Omega} F(x/a)\, dG(a) \text{ with } \Omega = [0, \infty). \tag{2.1.2}$$

This distribution is called scale mixture distribution. Particularly, when the distribution F is a normal distribution with mean 0 and variance σ^2, then (2.1.2) can be written as

$$H_G(x) = \int_0^{\infty} \Phi\left(x/\sigma\sqrt{a}\right) dG(a), \tag{2.1.3}$$

which is called scale mixture of normal distribution.

Every random variable X with scale mixture of normal distribution has the following stochastic representation

$$X =_d A^{1/2} Z, \tag{2.1.4}$$

where A is a positive random variable associated with distribution function $G(a)$, Z is the standard normal variate independent of A. The mixed variable X has the distribution function $H_G(x)$ in (2.1.3).

2.2 Properties of Scale Mixtures of Normal Distributions

Let $\mathcal{F} = \left\{ F(x) = \int_0^{\infty} \Phi\left(x/\sigma\sqrt{a}\right) dG(a),\ G \text{ is a cdf on } [0, \infty) \right\}$ be the collection of scale mixtures of normal distributions, then $\mathcal{F}$ has the following properties:

Property 2.1 (Density Function and Characteristic Function). *For every $F \in \mathcal{F}$ which is absolutely continuous, it has the density function as*

$$f(x) = \int_0^\infty \left(2\pi a\sigma^2\right)^{-1/2} e^{-\frac{x^2}{2a\sigma^2}} dG(a), \tag{2.2.1}$$

and its characteristic function is given by

$$\varphi(t) = \int_0^\infty e^{-x^2 a\sigma^2/2} dG(a). \tag{2.2.2}$$

According to Khinchine's theorem, X is unomodal *iff* $X =_d YU$, where U is uniformly distributed over [0, 1], and Y is a random variable independent of U, it is easy to show that scale mixture of normal distributions are unimodal, hence we have the following property:

Property 2.2 (Symmetry and Unimodality). *All density functions from* $\mathcal{F}$ *are symmetric and unimodal.*

Property 2.3 (Upper Bound for the Density). *Let* f *be the density function of a normal scale mixture random variable, for all* $a \geq -1$, *we have*

$$f(x) \leq k_a\, \mu_a / |x|^{1+a},$$

where $\mu_a = E\left(|X|^a\right)$ *and* $k_a = \left[(1+a)/e\right]^{(1+a)/2}\left[\Gamma\left((1+a)/2\right)2^{(1+a)/2}\right]^{-1}$.

Property 2.4 $\mathcal{F}$ *is closed under scale mixing operation and under addition (convolution)*

Proof. It is obvious that scale mixture of scale mixture of normal distribution is still scale mixture of normal. The closeness under convolution is also clear if one notices

$$\varphi_{X_1}(t)\varphi_{X2}(t) = \int_0^\infty e^{-t^2 a\sigma^2/2} d(G_1 * G_2)(a).$$

Note that $\forall\ X_1, X_2 \in \mathcal{F}$, $X_1 + X_2 =_d (A_1+A_2)^{1/2}Z$.

Property 2.5 (Identifiability). $\mathcal{F}$ *is identifiable, that is, for any* $X \in \mathcal{F}$ *if* $X =_d A_1^{1/2}Z =_d A_2^{1/2}Z$, *then* $A_1 =_d A_2$. *In other words, there is a one-to-one correspondence between* X *and* A.

Property 2.6 *$\mathcal{F}$ is closed under weak convergence. That is, if $F_n \to F$ as $n \to \infty$, then $F \in \mathcal{F}$. In other words, if $X_n \in \mathcal{F}$ converges to X weakly, then $X \in \mathcal{F}$* (Chandra, 1977).

Property 2.7 (Infinitely divisibility). *For any $F \in \mathcal{F}$, if its corresponding mixing distribution G is infinitely divisible, then F is also infinitely divisible.* (Feller, 1966). *Further more, when the corresponding mixing distribution G being completely monotonic, then $F \in \mathcal{F}$. is infinitely identifiable.*

Property 2.8 (Kurtosis). *If $X \in \mathcal{F}$ and its fourth moment exists (or equivalently the second moment of A exists), then the kurtosis is given by $3\left[1 + \left(\mathrm{var}(A)/E^2(A)\right)\right]$, which is greater than 3, the kurtosis for normal distribution.*

Property 2.9 (Moments inequality, Keilson and Steutel, 1974). *For any $X \in \mathcal{F}$ with finite second moment, we have*

$$\frac{Var(|X|)}{E^2|X|} \geq \frac{Var(|Z|)}{E^2|Z|} = \frac{\pi}{2} - 1,$$

and equality holds if and only if the mixing distribution is degenerated.

Proof: Applying Lyapunov inequality, the result follows.

Property 2.10 (Keilson and Steutel, 1974). *If the density function of $A^{1/2}$ is log-convex on $(0,\infty)$, then*

$$\frac{Var(|X|)}{E^2|X|} \geq 2\frac{Var(|Z|)}{E^2|Z|} + 1 = \pi - 1,$$

and if the density function of $A^{1/2}$ is log-concave on $(0,\infty)$, then

$$\frac{Var(|X|)}{E^2|X|} \leq 2\frac{Var(|Z|)}{E^2|Z|} + 1 = \pi - 1.$$

Property 2.11 (Rate of convergence to normality, Keilson and Steutel, 1974). *Define* $\rho(F_1,F_2)=\int_0^\infty (a-1)^2 |\mu_{A_1}(da)-\mu_{A_2}(da)|$ *for any* $F_1, F_2 \in \mathcal{F}$ *with* $E(A_1)=E(A_2)=1$, *then* ρ *defines a metric (distance) in subspace of* $\mathcal{F}$. *When* $A_2 =_d 1$, *i.e.* X_2 *is normal, then* ρ *is the distance measure to normality. Further more,* $\rho(F_X,\Phi)=\frac{\sigma_A^2}{\mu_A^2}$. *Thus, if the fourth moment of X exists, the degree of departure from normality is measured by square of coefficient of variation of the mixing variable.*

Recall that $h(x)$ is completely monotone ($c.m.$) if $(-1)^n h^{(n)}(x) \geq 0,\ \forall n$. The following theorem gives the necessary and sufficient condition for a distribution to be scale mixture of normal distribution based on the complete monotonicity.

Theorem 2.1 (Characterization, Kelker, 1971). *A distribution F belongs to* $\mathcal{F}$ *if and only if its c.f.* $\varphi(t)$ *or its density* $f(x)$ *is even function, and* $\varphi(\sqrt{t})$ *or* $f(\sqrt{x})$ *is completely monotone* on $(0,\infty)$.

Based the above theorem, for a given arandom variable, if it has explicit functional form of *c.f.* or density, we can check if it is a scale mixture of normal. Some special cases with particular mixing schemes are listed in the following corrollary:

Corollary 2.1 *The set of scale mixture distributions* $\mathcal{F}$ *contains the following symmetric distributions*

i) *Symmetric a-Stable distributions (SaS), when A is a positive a/2 stable (Feller, 1966).*

ii) *Laplace distribution with mean 0, when A/2 has exponential distribution.*

iii) *Student's t distribution, when A has the inverted gamma distribution.*

Tthe exponential power distributions (EPD) with $\mu=0$.

iv) *The logistic distribution with* $\mu=0$, *when* $A=(2K)^2$, *and K is the asymptotic Kolmogrov distance statistic (Andrew and Mallows 1974). Note that* $A=2(2K^2)=2\sum_{j=1}^{\infty} W_j/j^2$, *where* $W_1, W_2, \cdots$ *are iid exponential variables (Watson, 1961).*

The following two theorems are from Basu (1996).

Theorem 2.2 *Fix the 100p-th percentile* ($\frac{1}{2} < p < 1$)

$\sigma^2) = \{F \in \mathcal{F}: F(Q) = p \text{ and } Var(X) = \sigma^2\}$,

i) *If* $(Q/\sigma) \leq 1.1906$, *then the class* $\mathcal{F}(p, Q; \sigma^2)$ *is n*

ii) *If* $(Q/\sigma) > 1.1906$, *then the class* $\mathcal{F}(p, Q; \sigma^2)$ *is n*

As an application, suppose we obtain a set of dat tile=1.0. We try to fit a scale mixture of normal distr p=0.75, Q=1.0. Since $Q/\sigma < 1.1906$, $\Phi(Q/\sigma) > \Phi($ will not be able to find such a scale mixture of norma

Theorem 2.3 *Let* $Q_N = \sigma\Phi^{-1}(p)$, *i.e. let* Q_N *be the*

i) *If* $(Q_N/\sigma) \leq 1.1906$, *then the class* $\mathcal{F}(p, Q_N; \sigma^2)$ *is*

ii) *If* $(Q_N/\sigma) > 1.1906$, *then the class* $\mathcal{F}(p, Q_N; \sigma^2)$ *c*

Now we introduce the scale mixture of multivari $n \times 1$ random vector with mean **0**. We say **X** has a jo butions, if **X** has the following stochastic representati

$$\mathbf{X} =_d A^1$$

with A being a positive scalar random variable indep

positive definite covariance matrix of **G**. If we partit

tion Σ in conformance with **G** as $\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, then w

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} =_d$$

where $\mathbf{X}_1$ and $\mathbf{G}_1$ are $m \times 1$ vectors, with $1 \le m < n$. Most of the properties listed above can be easily extended to the multivariate case. In addition, there are some other properties for the multivariate scale mixture model.

Property 2.12 *The joint density function of* $\mathbf{X}$ *in* (2.2.3) *can be written* as

$$f(\mathbf{x}) = E_A\left[(2\pi A)^{-n/2}|\Sigma|^{1/2}\exp\left(-\mathbf{x}'\Sigma^{-1}\mathbf{x}/2A\right)\right]$$

Property 2.13 *The marginal distribution of* $\mathbf{X}_1$ and $\mathbf{X}_2$ *in* (2.2.4) *are also normal scale mixtures, i.e.,*

$$\mathbf{X}_1 =_d A^{1/2}\mathbf{G}_1 \textit{ and } \mathbf{X}_2 =_d A^{1/2}\mathbf{G}_2.$$

Property 2.14 $E\left[A^p\right] \le \infty$ *iff* $E\left[\mathbf{X}^{2p}\right] \le \infty$. *When the 2pth moment exists, we have*

$$E\left[A^p\right] = E\left[\|\mathbf{X}\|_{\Sigma^{-1}}^{2p}\right] / E\left[\|\mathbf{G}\|_{\Sigma^{-1}}^{2p}\right] = E\left[\|\mathbf{X}\|_{\Sigma^{-1}}^{2p}\right]\Gamma(n/2)/2^p\Gamma(n/2+p), \textit{for } p \ge 1,$$

where $E\left[\|\mathbf{X}\|_{\Sigma^{-1}}^{2p}\right] := E\left[\left(\mathbf{X}'\Sigma^{-1}\mathbf{X}\right)^p\right]$.

This equality holds because that $\mathbf{G}'\Sigma^{-1}\mathbf{G} \sim \chi^2(n)$ and $E\left[\|\mathbf{G}\|_{\Sigma^{-1}}^{2p}\right] = \Gamma(n/2)/2^p\Gamma(n/2+p)$.

Property 2.15 *The quadratic form of* $\mathbf{X}'\Sigma^{-1}\mathbf{X}$ *has scale mixture of Chi-square distribution.*

This is because $\mathbf{X}'\Sigma^{-1}\mathbf{X} =_d A\mathbf{G}'\Sigma^{-1}\mathbf{G}$, and $\mathbf{G}'\Sigma^{-1}\mathbf{G} \sim \chi^2(n)$.

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CHAPTER 3

THE INTEGRAL REPRESENTATIONS OF THE CONDITIONAL VARIANCE FOR SCALE MIXTURES OF NORMAL DISTRIBUTIONS

3.1 Introduction

The distribution of an n-dimensional random vector (column) $\mathbf{X}$ is a scale mixture of a normal distribution if $\mathbf{X} :=_d A^{1/2}\mathbf{G}$, where A is a positive random variable independent of the n-dimensional Gaussian random (column) vector $\mathbf{G}$ with mean $\mathbf{0}$ and positive definite covariance matrix Σ.

Gupta and Huang (1981) characterized scale mixtures (variance mixtures) of normal distributions by showing an equivalence of this class and the complete monotonicity property on $(0,\infty)$. Bearing this property, it was found that this family includes the Cauchy, Laplace, student's t, symmetric stable (these were also found by Kelker 1971), logistic and double exponential distributions (Andrews and Mallows, 1974). Schoenberg (1938), Crawford (1977), and Miciewicz and Scheffer (1990) characterized this family by showing that if $\mathbf{X}$ ($\mathbf{X} \in \mathbf{R}^n, n \geq 2$) is scale mixture of multivariate normal distribution, then its characteristic function, $\varphi_{\mathbf{X}}(\mathbf{t})$, $\mathbf{t} \in \mathbf{R}^n$, has the following representation: $\varphi_{\mathbf{X}}(\mathbf{t}) = \psi(\|\mathbf{t}^2\|)$, where $\|\cdot\|$ denotes the Euclidean distance, and ψ is some function on $(0,\infty)$. It should be added here that the family discussed by Schoenberg (1938), Crawford (1977), and Miciewicz and Scheffer (1990) is much broader than the family of scale mixtures of normal distributions. Keilson and Steutel (1974) characterized this family in terms of moment existence. It can be shown that $E[A^p] < \infty$, if and only if $E[\|\mathbf{X}\|^p] < \infty$ for some $p>0$. For example, if A is distributed as gamma, or beta or uniform then $E\left[\|\mathbf{X}\|^p\right] < \infty$, $\forall\, p > 0$. However, if A is totally right skewed $\alpha/2$-stable, $0<\alpha<2$, with Laplace transform $E\left[\exp(-uA)\right] = \exp(-u^{\alpha/2})$, $u \ge 0$ then $E\left[A^p\right] < \infty$, if and only if $p < \alpha/2$. In this case, $\mathbf{X}$ has a multivariate symmetric α-stable distribution, and $E\left[\prod_{i=1}^{n} |X_i|^{p_i}\right] < \infty$, for $p_i \ge 0$, $i=1, \ldots, n$, and $\sum_{i=1}^{n} p_i = p < \alpha$, see Samorodnitsky and Taqqu (1990). Thus, their second moment is always infinite and so is their first absolute moment when $0 < \alpha \le 1$.

Here, we are interested in conditional variances, and these may be finite even when their unconditional counterparts are infinite. For $1 \le m < n$, we will write $\mathbf{X}=(\mathbf{X}_1,\mathbf{X}_2)$, $\mathbf{G} = (\mathbf{G}_1,\mathbf{G}_2)$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where $\mathbf{X}_1$ and $\mathbf{G}_1$ are m-dimensional and Σ_{11} is $m \times m$-dimensional, i.e., Σ_{11} is the covariance matrix of $\mathbf{G}_1$, etc. The conditional distribution of $\mathbf{G}_2$ given $\mathbf{G}_1$ is normal with mean $\Sigma_{21}\Sigma_{11}^{-1}\mathbf{G}_1$ and covariance matrix $\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, i.e., the conditional mean of $\mathbf{G}_2$ given $\mathbf{G}_1$ depends linearly on $\mathbf{G}_1$ and the conditional variance-covariance of $\mathbf{G}_2$ given $\mathbf{G}_1$ is constant (degenerate, non random) and does not depend on the value of $\mathbf{G}_1$:

$$E\left[\mathbf{G}_2 | \mathbf{G}_1\right] = \Sigma_{21}\Sigma_{11}^{-1}\mathbf{G}_1, \qquad Cov(\mathbf{G}_2|\mathbf{G}_1) = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} := \Sigma_{2|1}. \tag{3.1.1}$$

This is the archetypical homoscedastic example, where regressions are linear and conditional variances constant. However, if $\mathbf{X}$ is scale mixture of multivariate normal, then Hardin (1982) showed that if $\mathbf{X} \in L^2(\Omega,\mathrm{F},\mathrm{P})$, $\mathbf{X}$ has the linear regression property. In fact, he showed that the linear regression property is equivalent to spherically generated processes, to which scale mixtures of multivariate normal belong. He continued by showing that if $\mathbf{X}$ is $S\alpha S$, $0 < \alpha < 2$, and $\dim(sp(\mathbf{X})) \ge 3$, then the linear regression property and sub-Gaussian are equivalent statements. This property is exactly the same as in the normal theory. The disagreement with the normal theory occurs when one looks at how the conditional variance-covariance behaves. It will be shown that scale mixtures of normal distributions do not have constant conditional variances, so they provide heteroscedastic examples, and we will examine these non-linear conditional functions.

This chapter is structured as follows. Section 3.2 presents the main results with their proofs. Section 3.3 demonstrates how to apply some of these results to uniform and stable cases. Section 3.4 gives the proofs of some of the secondary results. The auxiliary results are displayed in Section 3.5.

3.2 The Results

Our first result shows that the conditional second moment of each component of $\mathbf{X}_2$ given $\mathbf{X}_1$ is always finite when the dimensionality of $\mathbf{X}_1$ is two or more. Furthermore, we find a necessary and sufficient condition when $\mathbf{X}_1$ is univariate, and we express the conditional covariance matrix of $\mathbf{X}_2$ given $\mathbf{X}_1$ (under appropriate conditions) in terms of the distribution and the Laplace transform of A.

Theorem 3.1. *I. The conditional second moment of the components of $\mathbf{X}_2$ given $\mathbf{X}_1$ is finite a.s. always when $m \geq 2$ and if and only if $E\left[A^{1/2}\right] < \infty$ when $m=1$.*

II. If $m \geq 2$, or if $m=1$ and $E\left[A^{1/2}\right] < \infty$, then

$$Cov(\mathbf{X}_2|\mathbf{X}_1) = \Sigma_{2|1} S^2_{A,m}\left(\left(\mathbf{X}_1^{'}\Sigma_{11}^{-1}\mathbf{X}_1\right)^{1/2}\right) \quad a.s., \tag{3.2.1}$$

where

$$S^2_{A,m}(x) = \frac{\int_{[0,\infty)} u^{-m/2+1} \exp\left(-\frac{x^2}{2u}\right) dF_A(u)}{\int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{x^2}{2u}\right) dF_A(u)}, \quad x \geq 0. \tag{3.2.2}$$

III. If the Laplace transform L_A of A satisfies

$$\int_{[0,\infty)} u^{m/2-1} L_A(u) du < \infty \text{ and } \int_{[0,\infty)} u^{m/2-1} L'_A(u) du < \infty, \tag{3.2.3}$$

then (3.2.2) *holds and* $S^2_{A,m}(x)$, $x \geq 0$, *can be expressed as follows*

$$S^2_{A,1}(x) = \frac{-\int_0^\infty L'_A(r^2)\cos(\sqrt{2}xr)dr}{\int_0^\infty L_A(r^2)\cos(\sqrt{2}xr)dr}, \textit{and} \tag{3.2.4}$$

for $m \geq 2$,

$$S^2_{A,m}(x) = \frac{\int_0^\infty r^{m/2} L'_A(r^2) J_{\frac{m-2}{2}}(\sqrt{2}xr)dr}{\int_0^\infty r^{m/2} L_A(r^2) J_{\frac{m-2}{2}}(\sqrt{2}xr)dr}, \tag{3.2.5}$$

where $J_\nu(\cdot)$ *is the Bessel function of the first kind with* $\nu > 0$.

Proof. I. To demonstrate the proof of this theorem, we reiterate some of the classical results of normal theory. For simplicity of notation, it suffices to consider the case where $n=m+1$, so X_2, Σ_{22} are scalar. Then $E[X_2^2|\mathbf{X}_1] = E[E[AG_2^2|A, \mathbf{G}_1]|\mathbf{X}_1] = E[AE[G_2^2|\mathbf{G}_1]|\mathbf{X}_1]$, and since $E[G_2^2|\mathbf{G}_1] = \sigma_2^2 - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} + E^2[G_2|\mathbf{G}_1] = s_2^2 + (\Sigma_{21}\Sigma_{11}^{-1}\mathbf{G}_1)^2$, where $s_2^2 = \sigma_2^2 - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, we have

$$E[X_2^2|\mathbf{X}_1 = \mathbf{x}_1] = s_2^2 E[A|\mathbf{X}_1 = \mathbf{x}_1] + (\Sigma_{21}\Sigma_{11}^{-1}\mathbf{x}_1)^2. \tag{3.2.6}$$

It follows that $E\left[X_2^2|\mathbf{X}_1\right]<\infty$ a.s. if and only if $E\left[A|\mathbf{X}_1\right]<\infty$ a.s. and by Proposition 1 in Section 4, if and only if

$$\int_{(0,\infty)} u^{-m/2+1}\exp\left(-\frac{1}{2u}\mathbf{X}_1'\Sigma_{11}\mathbf{X}_1\right)dF_A(u)<\infty \quad a.s.. \tag{3.2.7}$$

Note that for each fixed value of $\mathbf{X}_1$, the integrand is a continuous function of u over $(0,\infty)$, and tends to 0 as $u\downarrow 0$ and as $u\uparrow\infty$ if $m\geq 2$ and is bounded by $u^{1/2}$ if $m=1$. Hence the conditional second moment is finite when $m\geq 2$ and when $m=1$ is finite if and only if

$$\int_0^\infty u^{1/2}dF_A(u)<\infty \text{ or } E\left[A^{1/2}\right]<\infty.$$

II. We have

$$E\left[\mathbf{X}_2\mathbf{X}_2'|\mathbf{X}_1\right]=E\left[E\left[A\mathbf{G}_2\mathbf{G}_2'|A,\mathbf{G}_1\right]|\mathbf{X}_1\right]=E\left[AE\left[\mathbf{G}_2\mathbf{G}_2'|\mathbf{G}_1\right]|\mathbf{X}_1\right],$$

and since

$$E\left[\mathbf{G}_2\mathbf{G}_2'|\mathbf{G}_1\right]=\Sigma_{2|1}+E\left[\mathbf{G}_2|\mathbf{G}_1\right]E\left[\mathbf{G}_2'|\mathbf{G}_1\right]=\Sigma_{2|1}+\Sigma_{21}\Sigma_{11}^{-1}\mathbf{G}_1\mathbf{G}_1'\Sigma_{11}^{-1}\Sigma_{21}', \tag{3.2.8}$$

and using the conditional expectation it follows that,

$$E\left[\mathbf{X}_2\mathbf{X}_2'|\mathbf{X}_1\right]=\Sigma_{2|1}E\left[A|\mathbf{X}_1\right]+\Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1\mathbf{X}_1'\Sigma_{11}^{-1}\Sigma_{21}'=\Sigma_{2|1}E\left[A|\mathbf{X}_1\right]+E\left[\mathbf{X}_2|\mathbf{X}_1\right]E\left[\mathbf{X}_2'|\mathbf{X}_1\right].$$

Thus the covariance is given by

$$\mathrm{Cov}(\mathbf{X}_2|\mathbf{X}_1)=E\left[\mathbf{X}_2\mathbf{X}_2'|\mathbf{X}_1\right]-E\left[\mathbf{X}_2|\mathbf{X}_1\right]E\left[\mathbf{X}_2'|\mathbf{X}_1\right]=\Sigma_{2|1}E\left[A|\mathbf{X}_1\right],$$

and by Proposition 1, $E\left[A|\mathbf{X}_1\right]=S_{A,m}^2\left(\left(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1\right)^{1/2}\right)$ with $S_{A,m}^2(x)$ as in Theorem 3.1.*II*.

III. For every $u\geq 0$ and (column) vector $\mathbf{t}\in\mathbf{R}^m$, we have

$$E\left[\exp(-uA+i\mathbf{t}'\mathbf{X}_1)\right]=E\left[E\left[\exp\left(-uA+iA^{1/2}\mathbf{t}'\mathbf{G}_1\right)\middle|A\right]\right]$$

$$=E\left[\exp\left(-uA-\frac{1}{2}A\mathbf{t}'\Sigma_{11}\mathbf{t}\right)\right]=L_A\left(u+\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right). \qquad (3.2.9)$$

Putting u=0 we obtain

$$E\left[\exp(i\mathbf{t}'\mathbf{X}_1)\right]=L_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right),$$

and since the right hand side is an integrable function of t over $\mathbf{R}^m$, in view of (3.2.4) we obtain

$$\int_{\mathbf{R}^m}L_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right)d\mathbf{t}=\left(\det\Sigma_{11}\right)^{-1/2}\int_{\mathbf{R}^m}L_A\left(\frac{1}{2}\mathbf{s}'\mathbf{s}\right)d\mathbf{s},\qquad\left(\mathbf{s}=\Sigma_{11}^{1/2}\mathbf{t}\right)$$

$$=\text{const}\int_0^\infty L_A\left(\frac{1}{2}r^2\right)r^{m-1}dr\qquad\text{(in polar coordinates)}$$

$$=\text{const}\int_0^\infty L_A(u)u^{m/2-1}du<\infty. \qquad (3.2.10)$$

By the inversion of the Fourier transform, we conclude that

$$f_{\mathbf{X}_1}(\mathbf{x}_1)=\frac{1}{(2\pi)^m}\int_{\mathbf{R}^m}e^{-i\mathbf{t}'\mathbf{x}_1}L_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right)d\mathbf{t}. \qquad (3.2.11)$$

Now differentiating both sides of (3.2.9) with respect to $u>0$, we obtain

$$-E\left[E\left[Ae^{-uA}\middle|\mathbf{X}_1=\mathbf{x}_1\right]e^{i\mathbf{t}'\mathbf{x}_1}\right]=L_A'\left(u+\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right).$$

Since $L_A(\cdot)$ is completely monotone on $(0,\infty)$, i.e., $(-1)^n L_A^{(n)}(u)\geq 0$, for $u>0$, it follows that $-L_A'$ $\left(u+\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right)\leq -L_A'\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right)\in\mathbf{L}^1(\mathbf{R}^m)$. By (3.2.10) and (3.2.2), inversion of the Fourier transform yields

$$- E\left[Ae^{-uA} \middle| \mathbf{X}_1 = \mathbf{x}_1\right] f_{\mathbf{X}_1}(\mathbf{x}_1) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} L'_A\left(u + \frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}, \text{ a.e. in } \mathbf{x}_1 \in \mathbf{R}^m, \quad (3.2.12)$$

for each fixed $u > 0$. Since $f_{\mathbf{X}_1}(\mathbf{x}_1)$ and the right hand side are continuous functions of $\mathbf{x}_1$ by *III.*, and in (3.1.4) we consider the regular version of $E\left[Ae^{-uA} \middle| \mathbf{X}_1 = \mathbf{x}_1\right]$, which is defined by (3.2.12) for all $u>0$ and $\mathbf{x}_1 \in \mathbf{R}^m$. Now, letting $u \downarrow 0$ in (3.2.12) we obtain

$$E\left[A \middle| \mathbf{X}_1 = \mathbf{x}_1\right] f_{\mathbf{X}_1}(\mathbf{x}_1) = -\frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} L'_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}, \quad (3.2.13)$$

since the left hand side of (3.2.12) converges pointwise to the left hand side of (3.2.13), and likewise for their right hand side by dominated convergence theorem, since $L_A(u) = E\left[e^{-uA}\right]$ implies $L'_A(u) = -E\left[Ae^{-uA}\right]$ and for all $v > 0$, $-L'_A(u+v) = E\left[Ae^{-(u+v)A}\right] \to E\left[Ae^{-vA}\right] = -L'_A(v)$, as $u \downarrow 0$, and $L'_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right) \in \mathrm{L}^1(\mathbf{R}^m)$ by (3.1.4) and (3.2.10). From (3.2.11) and (3.2.13) we obtain

$$S^2_{A,m}\left(\left(\mathbf{x}_1'\Sigma_{11}^{-1}\mathbf{x}_1\right)^{1/2}\right) = E\left[A \middle| \mathbf{X}_1 = \mathbf{x}_1\right] = \frac{-\int_{\mathbf{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} L'_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}}{\int_{\mathbf{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} L_A\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}}. \quad (3.2.14)$$

We will now evaluate more explicitly the integrals appearing in the numerator and denominator. Putting $B = 2^{-1/2}\Sigma_{11}^{1/2}$ and $\mathbf{y} = B\mathbf{t}$, we have $\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t} = \mathbf{t}'B'B\mathbf{t} = \mathbf{y}'\mathbf{y} = \|\mathbf{y}\|^2$ and

$$F_m\left(\left(\mathbf{x}_1'\Sigma_{11}\mathbf{x}_1\right)^{1/2}\right) := \int_{\mathbf{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} f\left(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t} = (\det B)^{-1} \int_{\mathbf{R}^m} e^{-i\mathbf{x}_1'B^{-1}\mathbf{y}} f\left(\|\mathbf{y}\|^2\right) d\mathbf{y}. \quad (3.2.15)$$

Going to polar coordinates $\mathbf{y} = r\mathbf{s},\ r \ge 0,\ \mathbf{s} \in U_m = \{\mathbf{s} \in \mathbf{R}^m : \|\mathbf{s}\| = 1\}$, we have, with γ_m being surface measure on U_m,

$$F_m\left(\left(\mathbf{x}_1'\Sigma_{11}^{-1}\mathbf{x}_1\right)^{1/2}\right) = \left(\frac{1}{2}\det\Sigma_{11}\right)^{-1/2} \int_0^\infty f\left(r^2\right) r^{m-1} dr \int_{U_m} \gamma_m(d\mathbf{s}) e^{-ir\mathbf{x}_1'B^{-1}\mathbf{s}} .$$

Putting $\mathbf{y}_1 = rB^{-1}\mathbf{x}_1$ we have $\|\mathbf{y}_1\|^2 = r^2\mathbf{x}_1'B^{-1}B^{-1}\mathbf{x}_1 = 2r^2\mathbf{x}_1'\Sigma_{11}\mathbf{x}_1$, and for $m=1$

$$\int_{U_1} e^{-i\mathbf{y}_1'\mathbf{s}}\gamma_1(d\mathbf{s}) = \cos(\|\mathbf{y}_1\|),$$

and for $m \ge 2$

$$\int_{U_m} e^{-i\mathbf{y}_1'\mathbf{s}}\gamma_m(d\mathbf{s}) = \int_0^\pi e^{-i\|\mathbf{y}_1\|\cos\theta}(\sin\theta)^{m-2}d\theta = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{m-1}{2}\right)}{\left(\frac{\|\mathbf{y}_1\|}{2}\right)^{\frac{m-2}{2}}} J_{\frac{m-2}{2}}\left(\|\mathbf{y}_1\|\right),$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind with $\nu > 0$. It follows that

$$F_1\left(|x_1|\sigma_1^{-1}\right) = \left(\frac{1}{2}\sigma_1^2\right)^{-1/2} \int_0^\infty f\left(r^2\right)\cos\left(\sqrt{2}r|x_1|\sigma_1^{-1}\right)dr,$$

$$F_m\left((\mathbf{x}_1'\Sigma_{11}^{-1}\mathbf{x}_1)^{1/2}\right) = \frac{\left(\frac{1}{2}\det\Sigma_{11}\right)^{-1/2}\pi^{1/2}\Gamma\left(\frac{m-1}{2}\right)}{\left(\frac{1}{2}\mathbf{x}_1\Sigma_{11}^{-1}\mathbf{x}_1\right)^{\frac{m-2}{4}}} \int_0^\infty r^{\frac{m}{2}} f\left(r^2\right) J_{\frac{m-2}{2}}\left(\sqrt{2}r\left(\mathbf{x}_1'\Sigma_{11}^{-1}\mathbf{x}_1\right)^{1/2}\right)dr. \quad (3.2.16)$$

The final expression for $S_{A,m}^2(x)$ now follows from (3.2.14)-(3.2.16).

It is clear from (3.2.1) that the conditional variance-covariance of $\mathbf{X}_2$ given $\mathbf{X}_1$ is proportional to its Gaussian counterpart, the constant conditional covariance matrix of $\mathbf{G}_2$ given $\mathbf{G}_1$, times a function $S_{A,m}^2(\cdot)$, depending on the dimensionality m of $\mathbf{X}_1$ and the distribution of A and evaluated at

$\left(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1\right)^{1/2}$. Thus, the heteroscedasticity of all conditional variances and covariance have a common functional form determined by the "conditional standard deviation factor" $S_{A,m}(x)$.

The expression in (3.2.2) is useful for evaluation when the distribution function of A is known explicitly. When this is not the case, but its Laplace transform is explicitly known, then the expressions in (3.2.4)-(3.2.5) are useful as illustrated below for the stable case.

Condition (3.2.3) can be expressed in terms of moments, by using

$$\int_0^\infty u^{p-1} E\left[A^k e^{-uA}\right] du = E\left[A^k \int_0^\infty u^{p-1} e^{-uA} du\right] = E\left[A^{k-p}\right]\Gamma(p).$$

Thus, condition (3.2.3) is equivalent to

$$E\left[A^{-1/2}\right] < \infty \text{ and } E\left[A^{1/2}\right] < \infty \text{ for } m=1, \text{ and } E\left[A^{-m/2}\right] < \infty \text{ for } m \geq 2. \qquad (3.2.17)$$

A useful alternative expression for $S_{A,m}^2(x)$ can be obtained in terms of the marginal density of the first component of the random vector $\mathbf{X}_1$ under the conditions in part (c) of Theorem 1.

Corollary 3.1. *Let* $f_1(|x|/\sigma_{11})$ *be the density of the first component of the random vector* $\mathbf{X}$ *(i.e., the density of* $\mathbf{X}_1$ *when m=1) where* σ_{11}^2 *is the* (1,1) *element of the covariance matrix* Σ. *Under the condition in Theorem* 1.III., *or* (3.2.17), *we have for* $x>0$,

$$S_{A,1}^2(x) = \frac{\int_{x^2}^\infty f_1(u)du}{2f_1\left(x^2\right)} \qquad (3.2.18.i)$$

$$S_{A,2k+1}^2(x) = -\frac{f_1^{(k-1)}\left(x^2\right)}{2f_1^{(k)}\left(x^2\right)}, \quad k \geq 1 \qquad (3.2.18.ii)$$

$$S_{A,2}^2(x) = \frac{\int_0^\infty u^{1/2} f_1^{(1)}(x^2 + u)du}{\int_0^\infty u^{-1/2} f_1^{(1)}(x^2 + u)du} \tag{3.2.18.iii}$$

$$S_{A,2k+2}^2(x) = -\frac{1}{2}\frac{\int_0^\infty u^{-1/2} f_1^{(k)}(x^2 + u)du}{\int_0^\infty u^{-1/2} f_1^{(k+1)}(x^2 + u)du}, \quad k \geq 1. \tag{3.2.18.iv}$$

Proof. It is known that (Kelker, 1970) since $\mathbf{X}_1$ is scale mixture of Normal distribution, i.e., has a spherical distribution, then the density $f_{\mathbf{X}_1}$ can be expressed as $f_{\mathbf{X}_1}(\mathbf{x}_1) = c_m\, g_m\left((\mathbf{x}_1'\Sigma_{11}^{-1}\mathbf{x}_1)^{1/2}\right)$ for all $\mathbf{x}_1 \neq 0$, $m \geq 1$, where g_m is a function on $(0,\infty)$, and $c_m = (2\pi)^{-m/2}|\Sigma_{11}|^{-1/2}$. Clearly $(2\pi\sigma_{11})^{-1/2}$ $g_1(|x|/\sigma_{11})$ is the density of the first component of $\mathbf{X}_1$. Since the integrand in (3.2.7) vanishes at 0, and A is assumed nondegenerate: $P(A = 0) < 1$, we have $0 < g_m(x) < \infty$ for all $x > 0$ and $m \geq 1$. Thus

$$S_{A,m}^2(x) = \frac{g_{m-2}(x)}{g_m(x)}, \quad x > 0, \quad m \geq 1. \tag{3.2.19}$$

Note that since (3.2.17) is satisfied, $g_{m-2}(x)$ is continuously differentiable over $x > 0$ for $m \geq 1$, with

$$\frac{g_{m-2}'(x)}{g_m(x)} = \frac{-x\int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{x^2}{2u}\right) dF_A(u)}{g_m(x)} = -x. \tag{3.2.20}$$

It follows from (3.2.19) and (3.2.20) that for $x > 0$, $m \geq 1$ $\frac{\left[S_{A,m}^2(x) g_m(x)\right]'}{g_m(x)} = -x$, and thus $S_{A,m}^2(x) g_m(x) = \int_x^\infty u g_m(u) du$. Hence (3.2.19) can be expressed as follows:

$$S_{A,m}^2(x) = \frac{\int_x^\infty u g_m(u) du}{g_m(x)}, \quad x > 0, \quad m \geq 1,$$

which follows,

$$S_{A,m}^2(x) = \frac{\int_{x^2}^\infty g_m(u^{1/2}) du}{2 g_m(x)}. \tag{3.2.21}$$

We will now express all g_m's in terms of g_1. From the definition of g_m and (3.2.19), it follows

$$g_1^{(k)}(x^2) = \frac{(-1)^k}{2^k} g_{2k+1}(x^2), \text{ and } g_2^{(k)}(x^2) = \frac{(-1)^k}{2^k} g_{2k+2}(x^2), \quad k \geq 1,$$

and thus from (3.2.21)

$$S_{A,2k+1}^2(x) = -\frac{1}{2} \frac{g_1^{(k-1)}(x^2)}{g_1^{(k)}(x^2)}, \quad k \geq 1, \text{ and } S_{A,2k+2}^2(x) = -\frac{1}{2} \frac{g_2^{(k-1)}(x^2)}{g_2^{(k)}(x^2)}, \quad k \geq 1. \tag{3.2.22}$$

It is easily checked that

$$g_2(x^2) = -\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty u^{-1/2} g_1^{(1)}(x^2 + u) du, \tag{3.2.23}$$

from which it follows that

$$S_{A,2k+2}^2(x) = -\frac{1}{2} \frac{\int_0^\infty u^{-1/2} g_1^{(k)}(x^2 + u) du}{\int_0^\infty u^{-1/2} g_1^{(k+1)}(x^2 + u) du}, \quad k \geq 1. \tag{3.2.24}$$

Thus (3.2.22) and (3.2.24) imply (3.2.18.ii) and (3.2.18.iv). These expressions of g_m, $m \geq 2$, in terms of g_1, in the more general setup of spherical distributions, are derived in Zolotarev, p.286 (1981). Szablosky (1987) has obtained similar expressions for elliptically contoured measures.

Also, (3.2.18.i) follows directly from (3.2.21) for $m=1$ and (3.2.18.iii) follows from (3.2.21) and (3.2.23). Note that g is a functions of both m and the density of A. However, the subscript of A is omitted for easing the reading of the content, since this does not change for different values of m.

Corollary 3.1 ties with the methods of Zolatarev (1981) and Szablowski (1986, 1987). In their studies they evaluated elliptically contoured measures with respect to suitable chosen marginal densities or conditional variances and the distribution of $\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1$, which is the case here, where the conditional variance is expressed with respect to the first component of the vector $\mathbf{X}_1$.

We now consider in more detail the types of heteroscedasticity provided by this model by examining the universal standard deviation function $S_{A,m}(x)$. We first show that under assumptions even more restrictive than those in part (c) of Theorem 1, the value of $S_{A,m}(x)$ at x=0, as given by the expressions (3.2.2) or (3.2.4)-(3.2.5), exists and is finite, $S_{A,m}(x)$ is continuous, differentiable, and is approximately quadratic around zero.

Corollary 3.2. *If the equivalent assumptions* (3.2.3) *in Theorem* 1*.III. or* (3.2.17) *hold for m+2, then we have*

$$S_{A,m}(x)=S_{A,m}(0)+C_{A,m}x^2+o(x^2) \text{ as } x\downarrow 0, \tag{3.2.25}$$

where $0<S_{A,m}(0)<\infty$ *and* $S_{A,m}(0)$, $C_{A,m}$ *are given in terms of moments of A:* $\mu_{A,p}=E\left[A^p\right]$ *with* $-\infty<p<\infty$ *as follows*

$$S_{A,m}(0)=\frac{\mu_{A,-m/2+1}^{1/2}}{\mu_{A,-m/2}^{1/2}},\quad C_{A,m}=\frac{\mu_{A,-m/2+1}\mu_{A,-m/2-1}-\mu_{A,-m/2}^2}{4\mu_{A,-m/2}^{3/2}\mu_{A,-m/2+1}^{1/2}} \tag{3.2.26}$$

and in terms of the Laplace transform of A by using

$$\mu_{A,-k/2} = \frac{2M_k(L_A)}{\Gamma(k/2)}, \quad k = 1,2,\ldots,m+2 \qquad \mu_{A,1/2} = \frac{2}{\sqrt{\pi}} M_1(-L'_A), \tag{3.2.27}$$

where $M_k(f) = \int_0^\infty r^{k-1} f(r^2) dr.$

Proof. From (3.2.19) we can write

$$S_{A,m}^2(x) - S_{A,m}^2(0) = \frac{g_{m-2}(x)}{g_m(x)} - \frac{g_{m-2}(0)}{g_m(0)}$$

$$= \frac{[g_{m-2}(x) - g_{m-2}(0)]g_m(0) - g_{m-2}(0)[g_m(x) - g_m(0)]}{g_m(x)g_m(0)}$$

$$= \frac{1}{g_m(x)g_m(0)} \left\{ g_m(0) \int_0^x g'_{m-2}(y)dy - g_{m-2}(0) \int_0^x g'_m(y)dy \right\}$$

$$= \frac{1}{g_m(x)g_m(0)} \left\{ -g_m(0) \int_0^x y g_m(y)dy + g_{m-2}(0) \int_0^x y g_{m+2}(y)dy \right\}$$

and using $\lim_{x\downarrow 0} \frac{1}{x^2} \int_0^x y g_m(y)dy = \lim_{x\downarrow 0} \frac{x g_m(x)}{2x} = \frac{1}{2} g_m(0)$ we obtain

$$\lim_{x\downarrow 0} \frac{1}{x^2}\left[S_{A,m}^2(x) - S_{A,m}^2(0)\right] = \frac{1}{2g_m^2(0)} \left\{ g_{m-2}(0) g_{m+2}(0) - g_m^2(0) \right\}.$$

But the left hand side is also $\lim_{x\downarrow 0} \frac{2}{x^2}\left[S_{A,m}(x) - S_{A,m}(0)\right] S_{A,m}(0)$, and thus

$$\lim_{x\downarrow 0} \frac{1}{x^2}\left[S_{A,m}(x) - S_{A,m}(0)\right] = \frac{g_{m-2}(0) g_{m+2}(0) - g_m^2(0)}{4 g_m^{3/2}(0) g_{m-2}^{1/2}(0)}.$$

The expression in Corollary 3.2 follows by using $g_m(0) = E\left[A^{-m/2}\right] = \mu_{A,-m/2}$.

To express $S_{A,m}(0)$ and $C_{A,m}$ in terms of the Laplace transform of A, we could use the expressions (3.2.4)-(3.2.5) instead of (3.2.2) and follow a similar line of argument, or equivalently, we could express the moments $\mu_{A,p}$ in terms of the Laplace transform L_A. This is accomplished by evaluating

$$M_m(L_A) = E\left[\int_0^\infty r^{m-1} e^{-r^2 A} dr\right] = \tfrac{1}{2} E\left[\int_0^\infty x^{m/2-1} e^{-xA} dx\right] = \tfrac{1}{2}\Gamma\left(\frac{m}{2}\right) E\left[A^{-m/2}\right],$$

for $m \geq 1$, so that $\mu_{A,-m/2} = 2M_m(L_A)/\Gamma(m/2)$, $m \geq 1$. This works for all the moments required in (3.2.25), with the exception of $\mu_{A,1/2}$ and $\mu_0 = 1$. This $\mu_{A,1/2}$, along with all $\mu_{A,-m/2}$, can be expressed using

$$\begin{aligned} M_m(-L_A') &= -\int_0^\infty r^{m-1} L_A'(r^2) dr = \int_0^\infty r^{m-1} E\left[A e^{-r^2 A}\right] dr \\ &= \frac{1}{2} E\left[A \int_0^\infty x^{m/2-1} e^{-xA} dx\right] = \frac{1}{2}\Gamma\left(\frac{m}{2}\right) E\left[A^{1-m/2}\right] = \frac{1}{2}\Gamma\left(\frac{m}{2}\right)\mu_{A,1-m/2}, \end{aligned}$$

for $m = 1$, leading to $\mu_{A,1/2} = \dfrac{2}{\sqrt{\pi}} M_1(-L_A')$.

When the assumption in Corollary 3.2 is not satisfied, i.e., when $E\left[A^{-m/2+1}\right] = \infty$, then a wide variety of (non-quadratic) asymptotic behavior at zero and at infinity is still possible. This results in a wide variety of heteroscedastic models illustrated in two examples in Section 3.3.

The following corollary describes the behavior of the factor $S_{A,m}(x)$ with respect to x for a given dimensionality. Furthermore, it shows how the higher the dimension we condition on, the lower the value of $S_{A,m}(x)$ becomes, for a given value of $x \in \mathbf{R}$.

Corollary 3.3. *If $F_A(0)=0$, then i) for any $m \geq 1$, $S_{A,m}(x)$ is non-decreasing in x>0, and ii) for any $x \geq 0$, $S_{A,m}(x)$ is non-increasing in $m \geq 1$.*

Proof. (i). Since

$$\frac{d}{dx} S_{A,m}^2(x) = \frac{d}{dx}\left(\frac{g_{m-2}(x)}{g_m(x)}\right) = \frac{g'_{m-2}(x)g_m(x) - g_{m-2}(x)g'_m(x)}{g_m^2(x)}$$

$$= \frac{x}{g_m^2(x)}\left\{g_{m-2}(x)g_{m+2}(x) - g_m^2(x)\right\},\ x>0.$$

It follows that $S_{A,m}(x)$ is non-decreasing if and only if the within the brackets quantity is greater or equal to zero, or

$$g_{m-2}(x)g_{m+2}(x) \geq g_m^2(x). \tag{3.2.28}$$

To show this, we proceed as follows. Let $A_x > 0$ denote the random variable associated with the random variable A, via the probability measure relationship

$$\nu_{A,x}(du) = \frac{u^{-(m+2)/2} e^{-\frac{x^2}{2u}} dF_A(u)}{\int_{[0,\infty)} u^{-(m+2)/2} e^{-\frac{x^2}{2u}} dF_A(u)}.$$

Assume $F_A(0) = 0$ to avoid some trivial difficulties.

Hence the necessary and sufficient condition (3.2.28) may be expressed in the form

$$\int_{[0,\infty)} u^2 \nu_{A,x}(du) \geq \left(\int_{[0,\infty)} u\nu_{A,x}(du)\right)^2, \text{ or equivalently, } E\left[A_x^2\right] \geq E\left[A_x\right]^2,$$

which is always true.

ii) To show that $S^2_{A,m}(x)$ is non-increasing with respect to $m = 1, 2, \cdots$ for fixed value of $x \in \mathbf{R}^+$, it is necessary and sufficient to show that $S^2_{A,m+1}(x) \le S^2_{A,m}(x)$, m=1, 2,..., for fixed x>0, or equivalently from (3.2.19), we need to show that $g_{m+1}(x)g_{m-2}(x) \ge g_m(x)g_{m-1}(x)$.

As in part (i), let $A_{x,1} > 0$ denote the random variable associated with the random variable A, and let $\theta_{A,x}(du)$ be modified version of $\nu_{A,x}(du)$ defined as follows.

$$\theta_{A,x}(du) = \frac{u^{-(m+1)/2} e^{-\frac{x^2}{2u}} dF_A(u)}{\int_{[0,\infty)} u^{-(m+1)/2} e^{-\frac{x^2}{2u}} dF_A(u)}.$$

Once again, the necessary and sufficient condition that the last inequality holds is to show that

$$\int_{[0,\infty)} u\theta_{A,x}(du) \int_{[0,\infty)} u^{1/2}\theta_{A,x}(du) \le \int_{[0,\infty)} u^{3/2}\theta_{A,x}(du),$$

or equivalently

$$E\left[A_{x,1}\right]E\left[A_{x,1}^{1/2}\right] \le E\left[A_{x,1}^{3/2}\right]. \qquad (3.2.29)$$

However the last inequality is always true, since $E\left[A_x^p\right]^{1/p}$ is a non decreasing function of p>0, and $E\left[A_{x,1}^{1/2}\right] \le E\left[A_{x,1}^{3/2}\right]^{1/3}$, and $E\left[A_{x,1}\right] \le E\left[A_{x,1}^{3/2}\right]^{2/3}$. Thus, by multiplying the last two inequalities, (3.2.29) is now evident.

This completes the proof of Corollary 3.3.

3.3. Examples

In this section we analyze the behavior of $S_{A,m}(x)$ in two specific cases, 1) when the random variable A is uniform and 2) when it is a positive stable. All the proofs of the following results will be deferred to Section 3.4.

3.3.1 UNIFORM SCALE MIXTURE. Here A is uniformly distributed over $[a, b]$, $0 \le a < b < \infty$.

First let $a>0$. Then, $E\left[A^p\right] < \infty$ for all $-\infty < p < \infty$, so by Corollary 3.1., all $S_{A,m}(x)$ are approximately quadratic around zero, i.e, (3.1.9) and (3.1.10) hold with

$$\mu_{A,p} = \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \text{ for all } p \in (-\infty, \infty) \text{ except } p = -1, \ \mu_{A,-1} = \frac{1}{b-a} \ln\left(\frac{b}{a}\right).$$

It is not hard to see that

$$a^{1/2} \le S_{A,m}(x) \le b^{1/2} \qquad \text{for all } m=1, 2, \ldots. \tag{3.3.1}$$

And at infinity all $S_m(x)$ tend to the same constant:

$$\lim_{x\to\infty} S_{A,m}(x) = b^{1/2}, \text{ for all } m=1, 2, \ldots. \tag{3.3.2}$$

Specifically, it is shown that for sufficiently large x

$$S_{A,m}(x) = \begin{cases} b^{1/2}\left(1 - \frac{b}{2x^2}\right) + o(x^{-2}) \text{ for } m \ne 4 \\ b^{1/2}\left(1 - \frac{b}{x^2}\right) + o(x^{-2}) \quad \text{for } m = 4. \end{cases} \tag{3.3.3}$$

Also from Corollary 3.3, $S_{A,m}(x)$, $m \ge 1$, increases from $S_{A,m}(0)$ to $b^{1/2}$.

Let $a=0$ Then, $E\left[A^p\right]<\infty$ only for $-1<p<\infty$ and thus $E\left[A^{-m/2-1}\right]=\infty$ for all $m \geq 1$, so Corollary 1 never applies. In this case, the limiting value of $S_{A,m}(x)$ at infinity vanishes except when $m=1$:

$$\lim_{x\to\infty} S_{A,m}(x)=\left(\frac{b}{3}\right)^{1/2}, \text{ and } \lim_{x\to\infty} S_{A,m}(x)=0, \quad m\geq 2. \tag{3.3.4}$$

The limiting value at zero is as (3.2.25). Around infinity, $S_{A,m}(x)$ is approximately linear for $m \geq 5$, whereas for smaller values of m it rises faster from its value at zero, the precise asymptotic expressions are presented as follows:

$$S_{A,1}(x)=\left(\frac{b}{3}\right)^{1/2}+o(x^2), \tag{3.3.5.i}$$

$$S_{A,2}(x)=b^{1/2}\left(\ln\frac{2b}{x^2}\right)^{-1/2}\left(1+\frac{\gamma}{2}\left(\ln\frac{2b}{x^2}\right)^{-1/2}\right)+o\left(\left(\ln\frac{1}{x}\right)^{3/2}\right), \tag{3.3.5.ii}$$

$$S_{A,3}(x)=\left(\frac{b}{2\pi}\right)^{1/2}x\left(1+\frac{x}{\sqrt{2b}}\right)+o(x^2), \tag{3.3.5.iii}$$

$$S_{A,4}(x)=\frac{x}{\sqrt{2}}\left(-\gamma+\ln\frac{2b}{x^2}\right)+o(x\ln\frac{1}{x}), \tag{3.3.5.iv}$$

$$S_{A,m}(x)=\frac{x}{\sqrt{m-4}}\left(1-\frac{x^{m-1}}{2(2b)^{m/2-2}\Gamma\left(\frac{m}{2}-1\right)}\right)+o(x^m), \quad m\geq 5, \tag{3..3.5.v}$$

where γ=0.57721, is the Euler's constant.

3.3.2 STABLE SCALE MIXTURE. Here $A \sim S_{\alpha/2}\left(\cos\left(\frac{\pi\alpha}{4}\right),1,0\right)$, $0<\alpha<2$, i.e., A is stable totally skewed to the right with $E\left[e^{-sA}\right]=e^{-s^{\alpha/2}}$. It will be shown that the scale factor, $S_{A,m}(x)$, which determines the shape of heteroscedasticity, can be expressed in an additive form with the dominant term being exactly the one we have achieved at infinity. On the other hand, the other term can be shown to explode to infinity with respect to "x", except at $\alpha=1$, which is constant. This result supports Cioczek-Georges and Taqqu's (1993) arguments for $m=1$.

It can be shown that the scale factor associated with the variance covariance matrix, $Cov(\mathbf{X}_2|\mathbf{X}_1)$, $\mathbf{X}_1 \in \mathbf{R}^m$, $m \geq 2$ has the following properties:

$$\lim_{x\to\infty} \frac{S_{A,m}^2(x)}{x^2} = \frac{1}{m+\alpha-2}. \tag{3.3.6}$$

The following result connects (3.3.6) by proving an additive relation, where the limiting term showing in (3.3.6) is one of the two terms.

$$S_{A,m}^2(x) = \frac{C(x;\, \alpha,\, m)}{4(m+\alpha-2)(m-1)} + \frac{x^2}{m+\alpha-2}, \tag{3.3.7}$$

where $C(x;\alpha,m) = \dfrac{\alpha^2(m-1)\int_{[0,\infty)} e^{-r^\alpha} r^{m/2+2(\alpha-1)} J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}{\int_{[0,\infty)} e^{-r^\alpha} r^{m/2} J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}$, and $J_\nu(x)$ is the Bessel function of the first kind. It is also shown that

$$\lim_{x\to\infty}\left|S_{A,m}^2(x) - \frac{x^2}{m+\alpha-2}\right| = \begin{cases} \infty & \text{for } \alpha \neq 1 \\ \dfrac{1}{4(m-1)} & \text{for } \alpha = 1. \end{cases} \tag{3.3.8}$$

REMARKS. When $\alpha = 1$, the functional form of $S_{A,m}^2(x)$, for $m \geq 2$, becomes a pure quadratic function. This was also noticed by Cioczek-Georges and Taqqu (1993) for $m=1$ when they studied the behavior of their stable conditional variance. Therefore, for $\alpha = 1$, the form is deduced to be

$$S^2_{A,m}(x) = \frac{1}{m-1}\left[x^2 + \frac{1}{4}\right],\ m \geq 2. \tag{3.3.9}$$

For the reason of completeness, we shall state the case $m=1$. This was approached by both Wu and Cambanis (1992) and Cioczek-Georges and Taqqu (1993) for the stable case. Here, it will be presented in the sub-Gaussian case. For $m=1$ the scale factor associated with the conditional variance, $Var(X_2|X_1)$, has the following properties:

$$\lim_{x\to\infty}\frac{S^2_{A,1}(x)}{x^2} = \frac{1}{\alpha-1},\ S^2_{A,1}(x) = \frac{C(x;\ \alpha,\ 1)}{2(\alpha-1)} + \frac{x^2}{\alpha-1},\text{ and }\lim_{x\to\infty}\left|S^2_{A,1}(x) - \frac{x^2}{\alpha-1}\right| = \infty, \tag{3.3.10}$$

where $C(x;\alpha,1) = \dfrac{\alpha\int_{[0,\infty)} e^{-r^\alpha} r^{2(\alpha-1)}\cos\left(\sqrt{2}xr\right)dr}{\int_{[0,\infty)} e^{-r^\alpha}\cos\left(\sqrt{2}xr\right)dr}$.

3.4 Proofs and Secondary Results

In the proof of Theorem 3.1, we use the following form of the regular conditional distribution of A given $\mathbf{X}_1$.

PROPOSITION 1. *For each non-negative measurable function* $g(\cdot)$ *we have*

$$E\left[g(A)|\mathbf{X}_1 = x_1\right] = \frac{\int_{[0,\infty)} g(u)u^{-m/2}\exp\left(-\frac{1}{2u}x_1{}'\Sigma_{11}^{-1}x_1\right)dF_A(u)}{\int_{[0,\infty)} u^{-m/2}\exp\left(-\frac{1}{2u}x_1{}'\Sigma_{11}^{-1}x_1\right)dF_A(u)}, \tag{3.4.1}$$

for almost every $\mathbf{X}_1 \in \mathbf{R}^m$, *where* $F_A(\cdot)$ *is the distribution function of* A.

Proof. It is well known that the joint density function of $\mathbf{X}_1 \in \mathbf{R}^m$ with $\mathbf{X}_1 =_d A^{1/2}\mathbf{G}_1$, where $\mathbf{G}_1$ is a centered Gaussian random vector with covariance matrix Σ_{11} is of the form:

$$f_{\mathbf{X}_1}(\mathbf{x}_1) = \frac{(\det \Sigma_{11})^{-1/2}}{(2\pi)^{m/2}} \int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{1}{2u}\mathbf{x}_1{}' \Sigma_{11}^{-1} \mathbf{x}_1\right) dF_A(u). \tag{3.4.2}$$

The rest of the proof is a simple consequence of the conditional expectation and the formula of the joint distribution of $\mathbf{X}_1$ and A.

Proof of (3.3.1). The proof of this follows by just noting that

$$S^2_{A,m}(x) = \frac{\frac{x^2}{2}\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-3} dy}{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2} dy} \begin{cases} \leq b \dfrac{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2} dy}{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2} dy} = b \\ \geq a \dfrac{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2} dy}{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2} dy} = a. \end{cases}$$

Proof of (3.3.2), (3.3.3), and (3.3.4). It is known (see e.g. Gradshteyn and Ryzhik, 1980, p.943) that for sufficiently large values of x and for any $a \in \mathbf{R}$, $x^{-(a-1)}e^x\Gamma(a,x) = 1 - \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2}$ $+ o(x^{-2})$, where $\Gamma(a,x) = \int_x^\infty e^{-y}y^{a-1}dy$, is the incomplete gamma function. Hence, for any m except $m = 4,\ 2$ we have

$$S^2_{A,m}(x)$$

$$= \left[\frac{b\left(\frac{x^2}{2b}\right)^{-\frac{m-6}{2}} e^{\frac{x^2}{2b}}\,\Gamma\left(\frac{m-4}{2}, \frac{x^2}{2b}\right)}{\left(\frac{x^2}{2b}\right)^{-\frac{m-4}{2}} e^{\frac{x^2}{2b}}} - \frac{a\left(\frac{x^2}{2a}\right)^{-\frac{m-6}{2}} e^{\frac{x^2}{2b}}\,\Gamma\left(\frac{m-4}{2}, \frac{x^2}{2b}\right)}{\left(\frac{x^2}{2a}\right)^{-\frac{m-4}{2}} e^{\frac{x^2}{2b}}}\right]\left[\Gamma\left(\frac{m-4}{2}, \frac{x^2}{2b}\right) - \Gamma\left(\frac{m-4}{2}, \frac{x^2}{2a}\right)\right]^{-1}$$

$$= \frac{b\left[1+\frac{b(m-6)}{x^2}+o\left(x^{-2}\right)\right]}{1+\frac{b(m-4)}{x^2}+\left(\frac{a}{b}\right)^{-\frac{m-4}{2}} e^{-\frac{x^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right)}+o(1)} - \frac{a\left[1+\frac{a(m-6)}{x^2}+o\left(x^{-2}\right)\right]}{\left(\frac{b}{a}\right)^{-\frac{m-4}{2}} e^{\frac{x^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right)}-1-\frac{a(m-4)}{x^2}+o(1)} = b\left(1-\frac{b}{x^2}\right)+o\left(x^2\right).$$

Taking the square root in both sides, the answer follows immediately.

PROOF FOR $\underline{m=4}$. Call $\Gamma(0,x)=\int_x^{\infty}\left(e^{-u}/u\right)du$. Hence, via Lemma 1

$$S_{A,4}^2(x)=\frac{b\,\frac{x^2}{2b}\,e^{\frac{x^2}{2b}}\,\Gamma\left(0,\frac{x^2}{2b}\right)}{e^{\frac{x^2}{2a}}\left[\Gamma\left(1,\frac{x^2}{2b}\right)-\Gamma\left(1,\frac{x^2}{2a}\right)\right]}+o\left(x^{-2}\right)=\frac{b\left(1-\frac{2b}{x^2}\right)}{1-e^{-\frac{x^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right)}}+o\left(x^{-2}\right)=b\left(1-\frac{2b}{x^2}\right)+o\left(x^{-2}\right).$$

This completes the proof for $m=4$.

Proof for $\underline{m=2}$. In exactly the same fashion as above, we note that

$$S_{A,2}^2(x)=\frac{\frac{x^2}{2}\left[\Gamma\left(-1,\frac{x^2}{2b}\right)-\Gamma\left(-1,\frac{x^2}{2a}\right)\right]}{\Gamma\left(0,\frac{x^2}{2b}\right)-\Gamma\left(0,\frac{x^2}{2a}\right)}=\frac{b\left(1+\frac{b}{x^2}+o\left(x^{-2}\right)\right)}{1+\frac{2b^2}{x^2}+o\left(x^{-2}\right)}=b\left(1-\frac{b}{x^2}\right)+o\left(x^{-2}\right).$$

Proof of (3.3.5i) If m=1, it can be seen that $S_{A,1}^2(x)=\dfrac{\frac{x^2}{2}\int_{x^2/2b}^{\infty}\frac{e^{-y}}{y^{5/2}}dy}{\int_{x^2/2b}^{\infty}\frac{e^{-y}}{y^{3/2}}dy}$. Now using integration by parts we have that $\int_x^{\infty}\frac{e^{-y}}{y^{a+1}}dy=\frac{1}{a}\left[\frac{e^{-x}}{x^a}-\int_x^{\infty}e^{-y}y^{-a}dy\right]$ and Lemma 3.2, it follows that the expression above may be written

$$S_{A,1}^2(x)=\frac{2b}{3}\left\{\frac{1}{\left(\frac{x^2}{2b}\right)^{1/2}e^{x^2/2b}\int_{x^2/2b}^{\infty}\frac{e^{-y}}{y^{1/2}}dy}-\frac{x^2}{2b}\right\}=\frac{b}{3}\left\{\frac{1}{1-\frac{x^2}{b}+o\left(x^2\right)}-\frac{x^2}{b}\right\}=\frac{b}{3}\left\{1+o\left(x^2\right)\right\}.$$

Proof of (3.3.5.ii) Repeating the same arguments, we may also have that

$$S_{A,2}^2(x)=\frac{\frac{x^2}{2}\int_{x^2/2b}^{\infty}y^{-2}e^{-y}dy}{\int_{x^2/2b}^{\infty}y^{-1}e^{-y}dy}=b\left\{\frac{1}{e^{x^2/2b}\int_{x^2/2b}^{\infty}e^{-y}y^{-1}dy}+\frac{x^2}{b}\right\}.$$

Thus, since

$$\int_x^{\infty}\frac{e^{-u}}{u}du=-\gamma+\ln\frac{1}{x}+x+o(x),\ x>0, \tag{3.4.3}$$

(Hardy 1949, p27), and $e^x=1+x+o(x)$, it implies that

$$S_{A,2}^2(x)=\frac{b}{\left(1+\frac{x^2}{2b}+o(x^2)\right)\ln(2b/x^2)\left(1-\frac{\gamma}{\ln(2b/x^2)}+o\left(\frac{1}{\ln(2b/x^2)}\right)\right)}+\frac{x^2}{2}$$

$$=\frac{b}{\ln(2b/x^2)}\left\{1+\frac{\gamma}{\ln(2b/x^2)}+o\left(\frac{1}{\ln(2b/x^2)}\right)\right\},$$

$$S_{A,4}^2(x)=\frac{x^2/2b\int_{x^2/2b}^{\infty}e^{-y}y^{-1}dy}{\int_{x^2/2b}^{\infty}e^{-y}dy}=\frac{x^2}{2}\left\{-\gamma+\ln\frac{2b}{x^2}\right\}+o\left(x^2\ln\frac{1}{x}\right).$$

Proof of (3.3.5.v). For $m\geq 5$, we just utilize (3.4.4),

$$S_{A,m}^2(x)=\frac{x^2/2b\int_{x^2/2b}^{\infty}e^{-y}y^{\frac{m-6}{2}}dy}{\int_{x^2/2b}^{\infty}e^{-y}y^{\frac{m-4}{2}}dy}=\frac{x^2}{2}\frac{\Gamma\left(\frac{m-4}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right)}\frac{1-\frac{\left(x^2/2b\right)^{\frac{m-4}{2}}}{\Gamma\left(\frac{m-2}{2}\right)}+o\left(x^{m-1}\right)}{1-\frac{\left(x^2/2b\right)^{\frac{m-2}{2}}}{\Gamma\left(\frac{m}{2}\right)}+o\left(x^m\right)}$$

$$= \frac{x^2}{m-4}\left\{1 - \frac{x^{m-1}}{(2b)^{m/2-2}\Gamma(m/2-1)} + o\left(x^{m-1}\right)\right\}.$$

This completes the proof of (3.5.5.v).

In establishing Theorem 3.2, we are aided by using some ideas from the Tauberian Theorem found in Samorodnisky and Taqqu (1994). We incorporate relations and identities given in Cambanis and Fotopoulos (1994), and we utilize various properties of the Bessel family. We continue by first resolving Theorem 3.1 and then Lemma 3.3.

Proof of (3.3.6) Since the choice of A is such that $A \sim S_{a\,2}(\sigma,\ 1,\ 0)$, $0 < a < 2$, $\sigma > 0$, we have that

$$P(A > x) \sim \frac{\sigma^{a\,2}}{\Gamma\left(1-\frac{a}{2}\right)\cos\frac{\pi a}{4}} x^{-a\,2} = c_{\sigma,a} x^{-a\,2} \text{ as } x \to \infty. \tag{3.4.5}$$

At this point we are interested to know the behavior of $g_m(x)$ as $x \to \infty$ occurred in (3.3.14) with the scalar being stable, and consequently to determine the behavior of $S^2_{A,m}(x)$ for large arguments of x. We shall cover both cases $m \geq 2$ with $a \in (0,\ 2)$, and $m = 1$ with $a \in (1,\ 2)$. Using integration by parts, it follows that

$$g_m(x) = -\int_{[0,\,\infty)} u^{-\frac{m}{2}} e^{-\frac{x^2}{2u}} dP(A > u)$$

$$= -u^{-\frac{m}{2}} e^{-\frac{x^2}{2u}} P(A > u)\Big|_0^{\infty} + \int_{[0,\,\infty)} P(A > u) d\left[u^{-\frac{m}{2}} e^{-\frac{x^2}{2u}}\right]$$

$$= \int_{[0,\,\infty)} e^{-\frac{x^2}{2u}}\left[-\frac{m}{2} u^{-\frac{m+2}{2}} + \frac{x^2}{2} u^{-\frac{m+4}{2}}\right] P(A > u) du$$

$$=\left(\tfrac{x^2}{2}\right)^{-\frac{m}{2}}\int_{[0,\infty)} e^{-\frac{x^2}{2u}}\left[-\tfrac{m}{2}\left(\tfrac{x^2}{2u}\right)^{\frac{m+2}{2}}\right]P\left(A>\tfrac{2u}{x^2}\tfrac{x^2}{2}\right)d\left(\tfrac{2u}{x^2}\right)$$

$$=\left(\tfrac{x^2}{2}\right)^{-\frac{m-2}{2}}\int_{[0,\infty)} e^{-y}\left[-\tfrac{m}{2}y^{\frac{m-2}{2}}+y^{\frac{m}{2}}\right]P\left(A>\tfrac{x^2}{2y}\right)dy \tag{3.4.6}$$

In connection (3.3.27), it follows that for $m\geq 2$, $a\in(0,\ 2)$ and $m=1$, $a\in(1,\ 2)$ and for $x\to\infty$,

$$\begin{aligned} g_m(x) &\sim c_{\sigma,a}\left(\frac{x^2}{2}\right)^{-\frac{m-2+a}{2}}\int_{[0,\infty)} e^{-y}\left\{-\frac{m}{2}y^{\frac{m+a-2}{2}}\right\}dy \\ &= c_{\sigma,a}\left(\frac{x^2}{2}\right)^{-\frac{m-2+a}{2}}\Gamma\left(\frac{m+a}{2}\right)\frac{a}{2}, \end{aligned} \tag{3.4.7}$$

which leads to

$$S_{A,m}^2(x)=\frac{g_{m-2}(x)}{g_m(x)}\sim\frac{c_{\sigma,a}\frac{a}{2}\left(\frac{x^2}{2}\right)^{-\frac{m-4+a}{2}}\Gamma\left(\frac{m-2+a}{2}\right)}{c_{\sigma,a}\frac{a}{2}\left(\frac{x^2}{2}\right)^{-\frac{m-2+a}{2}}\Gamma\left(\frac{m+a}{2}\right)}=\frac{x^2}{m+a-2}. \tag{3.4.8}$$

This completes the proof of part (3.3.6).

Remark. Obviously, if $m=1$ and $a\in(0,\ 1)$, then $E\left[A^{1/2}\right]=\int_{[0,\infty)}u^{1/2}\exp\left(-\frac{x^2}{2u}\right)dF_A(u)=\infty$.

This follows from the fact that $u^{1/2}P(A>u)\uparrow\infty$ as $u\to\infty$, this is true, because $P(A>u)\sim c_{\sigma,a}u^{\frac{1-a}{2}}\to\infty$, as $u\to\infty$ for $a\in(0,1)$. This concludes that $E\left[A^{1/2}\right]=\infty$.

Proof of (3.3.7). For simplicity, we set $a=\left(\frac{\sigma}{\cos\frac{\pi a}{4}}\right)^{2/a}=1$.

Call

$$A(x;\alpha,m)=\frac{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m+\alpha}{2}}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m}{2}}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}, \tag{3.4.9}$$

$$B(x;\alpha,m)=\alpha(m+\alpha-2)\tfrac{m}{2}\sqrt{2}x\frac{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m+\alpha-4}{2}}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m}{2}}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}$$

$$=(m+\alpha-2)m\sqrt{2}xS^{2}_{A,m}(x), \tag{3.4.10}$$

$$C(x;\alpha,m)=a^{2}\,\tfrac{m}{2}\sqrt{2}x\frac{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m+\alpha-4}{2}}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m}{2}}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)dr}. \tag{3.4.12}$$

From Lemma 3.3, we obtain that

$$(m+\alpha-2)m\sqrt{2}xS^{2}_{A,m}(x)=-A(x;\ \alpha,\ m)+B(x;\ \alpha,\ m)+\tfrac{m}{2}\left(\sqrt{2}x\right)^{3}$$

$$=C(x;\alpha,m)-B(x;\alpha,m)+B(x;\alpha,m)+\tfrac{m}{2}\left(\sqrt{2}x\right)^{3}=C(x;\ \alpha,\ m)+2(m-1)x^{2}. \tag{3.4.13}$$

This completes the proof of part (3.3.7).

Proof of (3.3.8). For convenience, we set $\lambda=\sqrt{2}x$. From (3.4.13), it follows that

$$\frac{m}{2\lambda}C(x;\alpha,m)=\alpha^{2}\frac{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m+4\alpha-4}{2}}J_{\frac{m-2}{2}}(\lambda r)dr}{\int_{[0,\infty)}e^{-r^{\alpha}}r^{\frac{m}{2}}J_{\frac{m-2}{2}}(\lambda r)dr}$$

$$=\frac{\alpha^{2}}{\lambda^{2(\alpha-1)}}\frac{\int_{[0,\infty)}e^{-\left(\frac{u}{\lambda}\right)^{\alpha}}u^{\frac{m+2\alpha-4}{2}}J_{\frac{m-2}{2}}(u)du}{\int_{[0,\infty)}e^{-\left(\frac{u}{\lambda}\right)^{\alpha}}u^{\frac{m}{2}}J_{\frac{m-2}{2}}(u)du}=\frac{\alpha^{2}}{\lambda^{2(\alpha-1)}}\frac{N(x;\ \alpha,\ m)}{D(x;\ \alpha,\ m)}. \tag{3.4.14}$$

We first examine $N(x;\alpha,m)$. It is clear that

$$N(x;\alpha,m) = \int_{[0,\,\Delta)} + \int_{[\Delta,\,\infty)} = I_1 + I_2\,, \text{ for } \Delta = \Delta(\lambda). \tag{3.4.15}$$

We take $\Delta/\lambda < 1$, $\Delta/\lambda \to 0$, as $\lambda \uparrow \infty$ and both Δ and λ tend to infinity. It can be checked that

$$I_1 = \frac{1}{\lambda}\int_{[0,\,\Delta)} \frac{e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} - 1}{1/\lambda} u^{\frac{m+4(\alpha-1)}{2}} J_{\frac{m-2}{2}}(u)du + \int_{[\Delta,\,\infty)} u^{\frac{m+4(\alpha-1)}{2}} J_{\frac{m-2}{2}}(u)du$$

$$\sim -\frac{1}{\lambda^{\alpha}} \int_{[0,\,\Delta)} u^{\frac{m+6\alpha-4}{2}} J_{\frac{m-2}{2}}(u)du + \int_{[\Delta,\,\infty)} u^{\frac{m+4(\alpha-1)}{2}} J_{\frac{m-2}{2}}(u)du\,, \tag{3.4.16}$$

since as $x \downarrow 0$ $\dfrac{e^{-x^{\alpha}} - 1}{x} = x^{\alpha-1} + O\left(x^{2\alpha-1}\right)$.

Obviously, the members on the right hand side of (3.4.16) are in the form of Lemma 3.5. From Lemmas 3.7 and 3.8, it can be seen that the dominant contribution of the right hand side of Lemma 3.6 is emanating from "$aJ_{\nu-1}(a)S_{\mu,\nu}(a)$".

From Lemma 3.6 and 3.8, we have that as $x \to \infty$

$$J_\nu(x) = \sqrt{\tfrac{2}{\pi x}\cos\left(x - \tfrac{2\nu+1}{4}\pi\right)} + o\left(x^{-1/2}\right), \text{ and } S_{\mu,\nu}(x) = x^{\mu-1} + O\left(x^{\mu-2}\right) \text{ for } p=1. \tag{3.4.17}$$

In conjunction with Lemma 3.5 and (3.4.16), (3.4.17) becomes

$$I_1 \sim \sqrt{\tfrac{2}{\pi}}\cos\left(\Delta - \tfrac{m-1}{4}\pi\right)\left[\frac{\Delta^{\frac{m-6\alpha-5}{2}}}{\lambda^{\alpha}} - \Delta^{\frac{m-6\alpha-5}{2}}\right]. \tag{3.4.18}$$

Next, we consider I_2. By Lemma 3.6

$$I_2 = \int_{[\Delta,\,\infty)} e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} u^{2(\alpha-1)} u^{\frac{m}{2}} J_{\frac{m-2}{2}}(u)du = \int_{[\Delta,\,\infty)} e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} u^{2(\alpha-1)} d\int_{[0,\,u)} y^{\frac{m}{2}} J_{\frac{m-2}{2}}(y)dy$$

$$= -e^{-\left(\frac{\Delta}{\lambda}\right)^{\alpha}} \Delta^{2(\alpha-1)+\frac{m}{2}} J_{\frac{m}{2}}(\Delta) - \int_{[\Delta,\infty)} u^{\frac{m}{2}} J_{\frac{m}{2}}(u) d\left[e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} u^{2(\alpha-1)}\right]$$

$$= -e^{-\left(\frac{\Delta}{\lambda}\right)^{\alpha}} \Delta^{2(\alpha-1)+\frac{m}{2}} J_{\frac{m}{2}}(\Delta) + \frac{a}{\lambda^{\alpha}} \int_{[\Delta,\infty)} e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} u^{3(\alpha-1)+\frac{m}{2}} J_{\frac{m}{2}}(u) du$$

$$= -2(\alpha-1) \int_{[\Delta,\infty)} e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} u^{2\alpha-3+\frac{m}{2}} J_{\frac{m}{2}}(u) du = I_{21} + \alpha I_{22} - 2(\alpha-1) I_{23}, \text{ say.} \tag{3.4.19}$$

In view of (3.4.16) and (3.4.17), we obtain

$$I_{21} \sim \sqrt{\frac{2}{\pi}} \cos\left(\Delta - \frac{m+1}{4}\pi\right) \Delta^{\frac{m+4a-5}{2}}. \tag{3.4.20}$$

To obtain I_{22}, some additional algebra is needed. From (3.4.16)

$$I_{22} = \sqrt{\frac{2}{\pi}} \lambda^{-\alpha} \int_{[\Delta,\infty)} e^{-\left(\frac{u}{\lambda}\right)^{\alpha}} u^{\frac{m-6a-7}{2}} du \sim \frac{1}{a} \sqrt{\frac{2}{\pi}} \frac{\lambda^{\frac{m-6a-5}{2}}}{\lambda^{\alpha}} \int_{[(\Delta/\lambda)^{\alpha},\infty)} y^{\frac{m-4a-5}{2a}} e^{-y} dy$$

$$\sim \frac{1}{a} \sqrt{\frac{2}{\pi}} \frac{\lambda^{\frac{m+6a-5}{2}}}{\lambda^{a}} e^{-\left(\frac{\Delta}{\lambda}\right)^{\alpha}} \left(\frac{\Delta}{\lambda}\right)^{\frac{m+4a-5}{2}} \sim \frac{1}{a} \sqrt{\frac{2}{\pi}} \Delta^{\frac{m+4a-5}{2}}. \tag{3.4.21}$$

In exactly the same way we continue for I_{23},

$$I_{23} \sim \frac{1}{a} \sqrt{\frac{2}{\pi}} \lambda^{\frac{m+4a-5}{2}} \int_{[(\Delta/\lambda)^{\alpha},\infty)} y^{\frac{m+2a-5}{2a}} e^{-y} dy \sim \frac{1}{a} \sqrt{\frac{2}{\pi}} \lambda^{a} \Delta^{\frac{m+2a-5}{2}}. \tag{3.4.22}$$

Combining (3.4.16), (3.4.18)-(3.4.22), (3.4.15) becomes

$$|N(\lambda; a, m)| = \sqrt{\frac{2}{\pi}} \Delta^{\frac{m-1}{2}} \left[c_1 \frac{\Delta^{3a-2}}{\lambda^{a}} + c_2 \Delta^{2(a-1)} + c_3 \lambda^{a} \Delta^{a-2}\right], \tag{3.4.23}$$

where c_1, c_2, and c_3 are positive suitable constants.

We proceed by investigating the behavior of the denominator.

$$D(\lambda;a,m) = \int_{[0,\Delta)} + \int_{[\Delta,\infty)} = I_1' + I_2', \text{ say.} \tag{3.4.24}$$

Using identical arguments as before and Lemma 3.6, we have that

$$I_1' = \frac{1}{\lambda^a}\int_{[0,\Delta)} \frac{e^{-\left(\frac{u}{\lambda}\right)^a}-1}{V_\lambda} u^{\frac{m}{2}} J_{\frac{m-2}{2}}(u)du + \int_{[0,\Delta)} u^{\frac{m}{2}} J_{\frac{m-2}{2}}(u)du$$

$$\sim \sqrt{\tfrac{2}{\pi}}\,\tfrac{1}{\lambda^a}\cos\left(\Delta - \tfrac{m-1}{4}\pi\right)\Delta^{\frac{m+2a-1}{2}} + \sqrt{\tfrac{2}{\pi}}\cos\left(\Delta - \tfrac{m+1}{4}\pi\right)\Delta^{\frac{m-1}{2}}. \tag{3.4.25}$$

Applying similar ideas as in (3.4.19), it follows that

$$I_2' = -e^{-\left(\frac{\Delta}{\lambda}\right)^a}\Delta^{\frac{m}{2}} J_{\frac{m}{2}}(\Delta) + \frac{a}{\lambda^a}\int_{[\Delta,\infty)} e^{-\left(\frac{u}{\lambda}\right)^a} u^{\frac{m+2a-2}{2}} J_{\frac{m}{2}}(u)du \sim -I_{21}' + aI_{22}', \text{ say.} \tag{3.4.26}$$

Clearly,

$$I_{21}' \sim \sqrt{\tfrac{2}{\pi}}\cos\left(\Delta - \tfrac{m+1}{4}\pi\right)\Delta^{\frac{m-1}{2}}, \tag{3.4.27}$$

and

$$I_{22}' \sim \frac{1}{\lambda^a}\int_{[\Delta,\infty)} e^{-\left(\frac{u}{\lambda}\right)^a} u^{\frac{m+2a-3}{2}} du \sim \sqrt{\tfrac{2}{\pi}}\cos\left(\Delta - \tfrac{m+1}{4}\pi\right)\Delta^{\frac{m-1}{2}}. \tag{3.4.28}$$

Combining (3.4.25)-(3.4.28), (3.4.24) becomes

$$D(\lambda;\alpha,m) \sim \sqrt{\tfrac{2}{\pi}}\Delta^{\frac{m-1}{2}}\left[\tfrac{1}{\lambda^a}\cos\left(\Delta - \tfrac{m-1}{4}\pi\right)\Delta^a + \alpha\right]. \tag{3.4.29}$$

In connection with (3.3.44) and (3.3.50), (3.3.35) becomes

$$\frac{m}{2\lambda}C(x;\alpha,m) \sim c_1\left(\tfrac{\Delta}{\lambda}\right)^{2(\alpha-1)} + c_2\left(\tfrac{\Delta}{\lambda}\right)^{\alpha-2} + c_3\left(\tfrac{\Delta}{\lambda}\right)^2, \tag{4.30}$$

where c_1, c_2, and c_3 are positive constants. This completes the proof of (3.3.8).

3.5 Auxiliary Results

LEMMA 3.1: *For sufficiently large x,*

$$e^x \int_x^\infty \frac{e^{-u}}{u} du = \frac{1}{x} \sum_{j=0}^m (-1)^j \frac{j!}{x^j} + o\left(x^{-m}\right).$$

Proof. Note that

$$e^x \int_x^\infty \frac{e^{-u}}{u} du = \int_0^\infty \frac{e^{-v}}{v+x} dv = \frac{1}{x} \int_0^\infty \frac{e^{-v}}{1+v/x} dv = \frac{1}{x}\left(\int_0^x + \int_x^\infty\right)$$

$$= \frac{1}{x} \sum_{j=0}^m (-1)^j \frac{1}{x^j} \int_0^x e^{-v} v^j dv + o\left(x^{-m}\right) + O\left(e^{-x}\right) = \frac{1}{x} \sum_{j=0}^m (-1)^j \frac{j!}{x^j} + o\left(x^{-m}\right).$$

This completes the proof of the Lemma 3.1.

LEMMA 3.2: *For $a<1$,*

$$x^a e^x \int_x^\infty \frac{e^{-y}}{y^{a+1}} dy = \frac{1}{a} - \frac{x}{a(1-a)} + o\left(x^2\right) \ as \ x \downarrow 0.$$

Proof. This is an outcome of a simple integration by parts arguments.

LEMMA 3.3. For any k=0, 1, 2, ... the following recurrent relations are true

i) $\left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^v J_v(r)\right) = r^{v-k} J_{v-k}(r)$, *and*

ii) $\left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{-v} J_v(r)\right) = (-1)^k r^{-(v+k)} J_{v+k}(r)$.

LEMMA 3.4. *Let* $I(\lambda;\ m,\ a) = \int_{[0,\infty)} e^{-r^a} r^{m+a-1} \int_{[0,\infty)} \cos(\lambda r\cos\theta)\sin^m\theta d\theta dr$. *Then*

i) $$a\frac{\left(\frac{\lambda}{2}\right)^{\frac{m}{2}}}{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)} I(\lambda;\ m,\ a) = \lambda\int_{[0,\infty)} e^{-r^a} r^{\frac{m}{2}} J_{\frac{m-2}{2}}(\lambda r)dr$$

ii) $$\lambda\frac{\left(\frac{\lambda}{2}\right)^{\frac{m}{2}}}{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)} I(\lambda;\ m,\ a) = (m+a-2)\int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-4}{2}} J_{\frac{m-2}{2}}(\lambda r)dr$$

$$-a\int_{[0,\infty)} e^{-r^a} r^{\frac{m+4(a-1)}{2}} J_{\frac{m-2}{2}}(\lambda r)dr.$$

Proof. i) Via Lemma 3.2 i), and a simple integration by parts, we proceed as follows.

$$a\frac{\left(\frac{\lambda}{2}\right)^{\frac{m}{2}}}{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)} I(\lambda;\ a,\ m) = a\int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-2}{2}} J_{\frac{m}{2}}(\lambda r)dr = -\int_{[0,\infty)} r^{\frac{m}{2}} J_{\frac{m}{2}}(\lambda r)de^{-r^a}$$

$$= \int_{[0,\infty)} e^{-r^a} r\left(\frac{1}{r}dr^{\frac{m}{2}} J_{\frac{m}{2}}(\lambda r)\right) = \lambda\int_{[0,\infty)} e^{-r^a} r^{\frac{m}{2}} J_{\frac{m}{2}}(\lambda r)dr. \tag{4.5.1}$$

This completes the proof of i).

ii) Using Lemma 3.2 ii), we have that

$$\lambda\frac{\left(\frac{\lambda}{2}\right)^{\frac{m}{2}}}{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)} I(\lambda;\ a,\ m) = \lambda\int_{[0,\infty)} e^{-r^a} r^{m+a-1}\lambda^{\frac{m}{2}}\left((\lambda r)^{-\frac{m}{2}} J_{\frac{m}{2}}(\lambda r)\right)dr \tag{4.5.2}$$

$$= -\int_{[0,\infty)} e^{-r^a} r^{m+a-2}\lambda^{\frac{m}{2}} \frac{d}{d\lambda r}\left((\lambda r)^{-\frac{m-2}{2}} J_{\frac{m-2}{2}}(\lambda r)\right)dr$$

$$= \frac{m-2}{2}\int_{[0,\infty)} e^{-r^a} r^{m+a-2}\lambda^{\frac{m}{2}}(\lambda r)^{-\frac{m}{2}} J_{\frac{m-2}{2}}(\lambda r)dr - \int_{[0,\infty)} e^{-r^a} r^{m+a-2}\lambda^{\frac{m}{2}} \frac{(\lambda r)^{-\frac{m-2}{2}}}{\lambda} dJ_{\frac{m-2}{2}}(\lambda r)$$

$$= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-4}{2}} J_{\frac{m-2}{2}}(\lambda r) dr - \int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-2}{2}} dJ_{\frac{m-2}{2}}(\lambda r)$$

$$= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-4}{2}} J_{\frac{m-2}{2}}(\lambda r) dr + \int_{[0,\infty)} J_{\frac{m-2}{2}}(\lambda r) d\left\{ e^{-r^a} r^{\frac{m+2a-2}{2}} \right\}$$

$$= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-4}{2}} J_{\frac{m-2}{2}}(\lambda r) dr - a \int_{[0,\infty)} e^{-r^a} r^{\frac{m+4(a-1)}{2}} J_{\frac{m-2}{2}}(\lambda r) dr$$

$$+ \left(\frac{m}{2} + a - 1 \right) A_{\frac{m}{2}}(\lambda) \int_{[0,\infty)} e^{-r^a} r^{m\,2\,+a-2} J_{\frac{m-2}{2}}(\lambda r) dr$$

$$= (m + a - 2) \int_{[0,\infty)} e^{-r^a} r^{\frac{m+2a-2}{}} J_{\frac{m-2}{2}}(\lambda r) dr - a \int_{[0,\infty)} e^{-r^a} r^{\frac{m+4(a-1)}{2}} J_{\frac{m-2}{22}}(\lambda r) dr .$$

This completes the proof of Lemma 3.3.

LEMMA 3.5. *(Gradsteyn and Ryzhik, 1980, p. 684 eq. 6.56.13). For a>0 and* $\mu + \nu > 0$*, the following is always true*

$$a^{\mu+1} \int_{[0,\,1)} x^{\mu} J_{\nu}(ax) dx = \int_{[0,\,a)} x^{\mu} J_{\nu}(x) dx$$

$$= (\nu + \mu - 1) a J_{\nu}(a) + S_{\mu-1,\nu-1}(a) - a J_{\nu-1}(a) S_{\mu,\nu}(a) + 2^{\mu} \frac{\Gamma\left(\frac{1+\mu+\nu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)},$$

where $S_{\mu,\nu}(x)$ *is Lommel's function.*

LEMMA 3.6. *(Gradsteyn and Ryzhik, 1980, p. 683 eq. 6.56.5). For ν>0, the following equality holds holds*

$$a^{\nu} \int_{[0,\,1)} x^{\nu} J_{\nu-1}(a,x) dx = \int_{[0,\,a)} x^{\nu} J_{\nu-1}(x) dx = a^{\nu} J_{\nu}(a) .$$

LEMMA 3.7. *(Abramowitz and Stegun, 1972, p. 364). When ν is fixed and $x \to \infty$,*

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{ P(\nu,x)\cos\chi - Q(\nu,x)\sin\chi \right\},$$

where $\chi = x - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi, \quad \mu = 4\nu^2,$

$$P(\nu,x) = 1 - \frac{(\mu-1)(\mu-9)}{2!(8x)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8x)^4} - \ldots,$$

and

$$Q(\nu,x) = \frac{\mu-1}{8x} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \ldots .$$

LEMMA 3.8. *(Gradsteyn and Ryzhik, 1980, p.986 eq. 8.576). If $\mu \pm \nu$ is not a positive odd integer, then*

$$S_{\mu,\nu}(x) = x^{\mu-1}\sum_{m=0}^{p-1} \frac{(-1)^m \Gamma\left(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\nu + m\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu + m\right)}{(x\ 2)^m \Gamma\left(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu\right)} + O\left(x^{\mu-2p}\right).$$

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CHAPTER 4

FORM OF THE CONDITIONAL VARIANCE FOR GAMMA MIXTURES OF NORMAL DISTRIBUTIONS

4.1 Introduction

In Chapter 3, we studied the behavior of the scale mixtures of multivariate normal distributions under conditioning. The mathematical expressions for the conditional variance of scale mixture of normal distributions are developed with integral representations in a rather general setup and in an abstract manner. The complexity of functional form of conditional variance-covariance often makes it hard to manage in general. However, if some additional information is available, say, if the distribution of the mixing variable is given, we may be able to derive the conditional variance in a explicit form. As an example, we discussed the asymptotic properties at both around the origin and for large arguments when the scale mixing variables are Uniform and α–stable in Chapter 3. Experiencing the richness of these special cases, we attempt here to give a complete picture of the gamma scale mixture of multivariate normal distributions under conditioning. Since the gamma family is a quite rich family, which includes a lot of important distributions and has many applications in statistical modeling, it is worthwhile to study the asymptotic behaviors of conditional variance for the gamma scale mixture of normal distributions. In contrast to the conditional variance of multivariate normal distribution, which is degenerate (non-random), the conditional variance of gamma scale mixture of multivariate normal distributions is non-constant. This chapter we focus on the investigation of conditional variance for a scale mixture of normal distribution with the mixing variable being Gamma. We show that the results are reduced to a simple function, which is related to the modified Bessel functions. We make no moment assumptions in our analysis.

This chapter is organized as follows. Basic definitions and general discussion are given in Section 4.2. Explicit formula for the conditional variance with various asymptotic results and expanded discussion on the invertability issue are given in Section 4.3. In Section 4.4 we provide all the proofs. Section 4.5 displays graphs of various combinations of parameters of the non-constant conditional standard deviation.

4.2 Background

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a (non-degenerate) random vector in $\mathbf{R}^n$ expressed by the stochastic representation $\mathbf{X} = A^{1/2}\mathbf{G}$, where A is $Gamma(1, \nu)$, $\nu > 0$ and $\mathbf{G}$ is multivariate normal with $E[\mathbf{G}] = 0$ and $Cov(\mathbf{G}) = \Sigma$, with Σ being a non-singular symmetric $n \times n$ matrix. It can be seen that $E[\mathbf{X}_2|\mathbf{X}_1]$ exists *a.s.* and $E[\mathbf{X}_2|\mathbf{X}_1] = \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1$ *a.s.* for $\mathbf{X}_1 \in \mathbf{R}^m$ and $m<n$, and Σ_{21} and Σ_{11} are $(n-m) \times m$ and $m \times m$ partition matrices of Σ, respectively. It was shown that $Cov(\mathbf{X}_2|\mathbf{X}_1) = E[A|\mathbf{X}_1]\Sigma_{2|1}$ *a.s.*, where $\Sigma_{2|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ and $E[A|\mathbf{X}_1 = x_1] = S_{A,m}^2\left(\left(x_1'\Sigma_{11}^{-1}x_1\right)^{1/2}\right)$ *a.e.* with $S_{A,m}^2(x)$, $x \geq 0$. It was also shown in Chapter 3 that if $m \geq 2$ or if $m = 1$ and $E\left[A^{1/2}\right] < \infty$, $S_{A,m}(x)$, $x>0$, is finite for fixed x, it is non-decreasing of $x \geq 0$, and for any $x \geq 0$, it is non-increasing for $m \geq 1$.

If A is $Gamma(1, \nu)$, $\nu > 0$, we shall provide the exact expression of $S_{A,m}(x)$, and we shall give its limiting behavior at both zero and infinity. We shall support our analysis with various graphs at various combinations at m and ν.

4.3 Development

We now investigate and discuss the functional form of $S_{A,m}(x)$, $x>0$, when the mixing variable is $Gamma(1,\nu)$, $\nu>0$. Our result presents a simple expression for $S_{A,m}(x)$, and we provide its limiting form at both zero and infinity.

Theorem 4.1 *Let* $\mathbf{X} = A^{1/2}\mathbf{G} \in \mathbf{R}^n$, $n \geq 2$ *be a scale mixture of normal distribution, with* $A \sim$ $Gamma(1,\nu)$, $\nu>0$.

I. The conditional second moment of the component $\mathbf{X}_2$ *given* $\mathbf{X}_1$ *is always finite, and it is given by*

$$Cov(\mathbf{X}_2|\mathbf{X}_1) = \Sigma_{2|1} S^2_{A,m}\left(\left(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1\right)^{\frac{1}{2}}\right), \quad a.s.,$$

where $S^2_{A,m}(x) = \dfrac{x}{\sqrt{2}}\,\dfrac{K_{\frac{m}{2}-\nu-1}\left(\sqrt{2}x\right)}{K_{\frac{m}{2}-\nu}\left(\sqrt{2}x\right)}$, $x>0$, *and* $K_\nu(\cdot)$ *is the modified Bessel function or Heine's function.*

II. If $x \downarrow 0$ *and if* $l = \frac{m}{2} - \nu - 1$, *then*

$$S_{A,m}(x)=\begin{cases}\dfrac{x}{\sqrt{2l}}+o(x), & \text{for } l>0,\\ \dfrac{x}{\sqrt{2}}\sqrt{\ln\left(\dfrac{1}{2x^2}\right)}+o\left(x\left(\ln\dfrac{1}{x^2}\right)^{1/2}\right), & \text{for } l=0,\\ \left(\dfrac{x}{2}\right)^{l+1}\sqrt{\dfrac{\Gamma(-l)}{\Gamma(l+1)}}+o\left(x^{l+1}\right), & \text{for } l<0 \text{ and } l+1>0,\\ \dfrac{1}{\sqrt{\ln\left(\dfrac{1}{2x^2}\right)}}+o\left(\dfrac{1}{\sqrt{\ln\dfrac{1}{x^2}}}\right), & \text{for } l+1=0,\\ \sqrt{-(l+1)}+o(1), & \text{for } l+1<0.\end{cases}$$

III. If $x\uparrow\infty$, and if $l=\frac{m}{2}-\nu-1$, then

$$S_{A,m}(x)=\frac{\sqrt{x}}{2^{1/4}}\left\{1-\frac{1}{2\sqrt{2}x}(l-1/2)\right\}+o\left(1/\sqrt{x}\right).$$

Discussion: In Chapter 3, it was shown that if the expression of the density of the scale variable is available, then the minimum conditions required for $S_{A,m}(x)$ to exist is only $E\left[A^{1/2}\right]<\infty$ for $m=1$. However, if the expression of the Laplace transform is available, then we need to check some integrability conditions. Since the Laplace transform for gamma function is known, it is of interest to see what conditions are needed such that (3.1.4) equals (3.1.6) and (3.1.7) in Chapter 3.

Since $\int_0^\pi \sin^{2\nu}\vartheta\cos(z\cos\vartheta)d\vartheta=\pi^{1/2}\Gamma(\nu+1/2)(z/2)^{-\nu}J_\nu(z)$, $\mathrm{Re}(\nu)>-\frac{1}{2}$, where $J_\nu(z)$ is the Bessel function of the first kind, it follows that (3.1.7) can be expressed as

$$S^2_{A,m}(x)=\frac{\int_0^\infty r^{m/2}L'_A(r^2)J_{\frac{m-2}{2}}(\sqrt{2}xr)dr}{\int_0^\infty r^{m/2}L_A(r^2)J_{\frac{m-2}{2}}(\sqrt{2}xr)dr},\ x\geq 0, \tag{4.3.1}$$

where $L_A(\cdot)$ is the Laplace transform of A.

Note that $L_A'(r^2) = \frac{dL_A(r^2)}{2rdr}$. Suppose that $A \sim Gamma(1, \nu)$, then $L_A(r^2) = (1 + r^2)^{-\nu}$. In light of the above remarks, equation (4.3.1) may now be written as

$$S^2_{A,m}(x) = \frac{\nu}{2} \frac{\int_0^\infty r^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(\sqrt{2}xr)(1 + r^2)^{-(\nu+1)} dr}{\int_0^\infty r^{m/2} J_{\frac{m-2}{2}}(\sqrt{2}xr)(1 + r^2)^{-\nu} dr}, \quad x>0. \tag{4.3.2}$$

It is known that $\int_0^\infty \frac{t^{\nu+1} J_\nu(\alpha t) dt}{(t^2 + \alpha^2)^{\mu+1}} = \frac{\alpha^\mu z^{\nu-1}}{z^\mu \Gamma(\mu + 1)} K_{\nu-\mu}(\alpha z)$ for $\alpha > 0$, $Re(z) > 0$, and $-1 < \mathrm{Re}(\nu) < 2\,\mathrm{Re}(\mu) + 3/2$ (see e.g., Abramowitz and Stegun, eq. 11.4.44, p. 488). This implies that

$$S^2_{A,m}(x) = \frac{x}{\sqrt{2}} \frac{K_{\frac{m}{2}-\nu-1}(\sqrt{2}x)}{K_{\frac{m}{2}-\nu}(\sqrt{2}x)}, \quad \text{for } 2 < m < 4\nu - 1,\ x > 0. \tag{4.3.3}$$

Next, we shall investigate whether $S_{A,m}(x)$ satisfies (4.3.3) for $m = 1$ and 2. For $m = 1$, we have that

$$S^2_{A,m}(x) = \frac{\frac{\nu}{2} \int_0^\infty \cos(\sqrt{2}xr)(1 + r^2)^{-\nu-1} dr}{\int_0^\infty \cos(\sqrt{2}xr)(1 + r^2)^{-\nu} dr}, \quad x>0. \tag{4.3.4}$$

Since $\int_0^\infty \frac{\cos(xt) dt}{(1 + t^2)^{\nu+1/2}} = \frac{\pi^{1/2} (x\ 2)^\nu}{\Gamma(\nu + 1/2)} K_\nu(x)$, for $\mathrm{Re}(\nu) > -\frac{1}{2}$ and $x > 0$ (see e.g., Abramowitz and Stegun, eq. 9.6.25, p. 376), the proof for $m = 1$ easily follows.

To see for $m = 2$, we observe that

$$S_{A,2}^2(x) = \frac{\nu}{2}\,\frac{\int_0^\infty rJ_0\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu-1}dr}{\int_0^\infty rJ_0\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu}dr},\quad x>0, \tag{4.3.5}$$

and the solution follows from eq. 11.4.44 in Abramowitz and Stegun. Therefore (4.3.3) holds for $1 \le m < 4\nu - 1,\ x > 0$.

To make sure that condition (3.1.5) agrees with $1 \le m < 4\nu - 1$, we consider the following. Since $u^{\frac{m}{2}-1}L_A(u)$ and $u^{\frac{m}{2}-1}L_A'(u) \in \mathbf{L}^1(0,\infty)$ it implies that $u^{\frac{m}{2}-1}/(1+u)^{\nu} \in \mathbf{L}^1(0,\infty)$. However, this function is integrable if and only if $0 \le m < 2\nu$, which agrees with what we found above. Specifically, we note that the range of m is contained to that we have shown in (3.3.2), (3.3.4) and (3.3.5), respectively.

In light of the discussion, it is worth noting that $S_{A,m}^2(x)$ may be expressed in a more revealing form. We shall present this result in a form of corollary.

Corollary 4.1. *For* $1 \le m < 4\nu - 1$, *and* $A \sim Gamma(1,\ \nu)$,

$$S_{A,m}^2(x) = \begin{cases} \dfrac{\int_0^\infty rJ_0\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+p-1}dr}{\int_0^\infty rJ_0\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+p}dr}, & \text{if } \dfrac{m}{2} = p \\[2ex] \dfrac{\int_0^\infty r^{1/2}J_{-1/2}\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+p-1}dr}{\int_0^\infty r^{1/2}J_{-1/2}\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+p}dr}, & \text{if } \dfrac{m}{2} = p - \dfrac{1}{2} \end{cases},\quad x>0, \tag{4.3.6}$$

for $p \in \mathbf{N}$.

Proof. Note that, if $2 \le m < 4\nu - 1$, then by continuously integrating by parts, one can obtain that

$$\int_0^\infty r^{m/2}J_{\frac{m-2}{2}}\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu}dr = \frac{x}{(\nu-1)\sqrt{2}}\int_0^\infty r^{\frac{m-2}{2}}J_{\frac{m-4}{2}}\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+1}dr$$

$$
=\begin{cases} \dfrac{\left(x\sqrt{2}\right)^{p-1}}{(\nu-1)\cdots(\nu-p)} \int_0^\infty rJ_0\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+p} dr, & \text{if } \dfrac{m}{2}=p \\ \dfrac{\left(x\sqrt{2}\right)^{p-1}}{(\nu-1)\cdots(\nu-p)} \int_0^\infty r^{1/2} J_{-1/2}\left(\sqrt{2}xr\right)\left(\sqrt{2}xr\right)\left(1+r^2\right)^{-\nu+p}, & \text{if } \dfrac{m}{2}=p-\frac{1}{2} \end{cases}, \quad x>0. \qquad (4.3.7)
$$

Substituting (4.3.7) into (4.3.2), the result follows immediately. The cases for $m = 1$ or 2 are treated separately. Again both of them reveal the same conclusion. This completes the proof of the corollary.

4.4 Proofs

Since $A \sim Gamma(1, \nu)$ and $E\left[A^p\right] < \infty$, for $p > -1$, (3.1.4) may be written as

$$
S_{A,m}^2(x) = \frac{\int_0^\infty u^{-m/2+\nu} \exp\left(-u-\frac{x^2}{2u}\right) du}{\int_0^\infty u^{-m/2+\nu-1} \exp\left(-u-\frac{x^2}{2u}\right) du}, \quad x>0. \qquad (4.4.1)
$$

It is known that

$$
\int_0^\infty u^{-m/2+\nu-1} \exp\left(-u-\frac{x^2}{2u}\right) du = 2\left(\frac{\sqrt{2}}{2}x\right)^{-\frac{m}{2}+\nu} K_{\frac{m}{2}-\nu}\left(\sqrt{2}x\right), \quad x>0 \qquad (4.4.2)
$$

(see e.g., Gradshteyn and Ryzhik, eq. 3.471.9, p. 340). In view of (4.4.2), the proof of part I of the Theorem 4.1 is now completed.

To show part II of the theorem, we utilize the following approximation

$$
K_\nu(x) = \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}x\right)^{-\nu} + o\left(x^{-\nu}\right), \quad \mathrm{Re}(\nu) > 0, \qquad (4.4.3)
$$

for sufficiently small arguments of $x > 0$, (see e.g., Abramowitz and Stegun, eq. 9.6.8, p. 375). Thus, setting $l = \frac{m}{2} - \nu - 1 > 0$, it can be noticed that

$$S_{A,m}^2(x) = \frac{x}{\sqrt{2}} \frac{K_l(\sqrt{2}x)}{K_{l+1}(\sqrt{2}x)} = \frac{x}{\sqrt{2}} \frac{\frac{1}{2}\Gamma(l)(x\sqrt{2})^{-l} + o(x^{-l})}{\frac{1}{2}\Gamma(l+1)(x\sqrt{2})^{-(l+1)} + o(x^{-(l+1)})} = \frac{x^2}{2l} + o(x^2).$$

If $l = 0$, we have that $K_0(z) = \ln z^{-1}$, as $z \downarrow 0$ (see e.g., Abramowitz and Stegun, eq. 9.6.8, p.375). Hence, we have that

$$S_{A,m}^2(x) = \frac{x}{\sqrt{2}} \frac{\ln(\sqrt{2}x) + o(\ln x^{-1})}{\frac{1}{2}(x\sqrt{2})^{-1} + o(x^{-1})} = \frac{x^2}{2}\ln((2x^2)^{-1}) + o(x^2 \ln x^{-2}).$$

If $l < 0$ and $l + 1 > 0$, then, since $K_{-\nu}(\cdot) = K_\nu(\cdot)$, we have that

$$S_{A,m}^2(x) = \frac{x}{\sqrt{2}} \frac{K_l(\sqrt{2}x)}{K_{l+1}(\sqrt{2}x)} = \frac{x}{\sqrt{2}} \frac{K_{-l}(\sqrt{2}x)}{K_{l+1}(\sqrt{2}x)}$$

$$= \frac{x}{\sqrt{2}} \frac{\frac{1}{2}\Gamma(-l)(x\sqrt{2})^{l} + o(x^{l})}{\frac{1}{2}\Gamma(l+1)(x\sqrt{2})^{-(l+1)} + o(x^{-(l+1)})} = \left(\frac{x}{\sqrt{2}}\right)^{2(l+1)} \frac{\Gamma(-l)}{\Gamma(l+1)} + o(x^{2(l+1)}). \quad (4.4.4)$$

If $l + 1 = 0$,

$$S_{A,m}^2(x) = \frac{x}{\sqrt{2}} \frac{K_1(\sqrt{2}x)}{K_0(\sqrt{2}x)} = \frac{x}{\sqrt{2}} \frac{\frac{1}{2}(x\sqrt{2})^{-1} + o(x^{-1})}{\ln((\sqrt{2}x)^{-1}) + \ln x^{-1}} = \frac{1}{\ln((2x^2)^{-1})} + o\left(\frac{1}{\ln x^{-2}}\right), \quad (4.4.5)$$

and if $l + 1 < 0$,

$$S^2_{A,m}(x) = \frac{x}{\sqrt{2}}\,\frac{K_{-l}(\sqrt{2}x)}{K_{-(l+1)}(\sqrt{2}x)} = \frac{x}{\sqrt{2}}\,\frac{\frac{1}{2}\Gamma(-l)(x\sqrt{2})^{l} + o(x^{l})}{\frac{1}{2}\Gamma(-(l+1))(x\sqrt{2})^{(l+1)} + o(x^{(l+1)})} = -(l+1) + o(1), \quad (4.4.6)$$

and thus to the conclusion of part II.

Finally, to show part III, we use the known approximation for large arguments,

$$K_\nu(z) = \sqrt{\frac{2\pi}{2z}}e^{-z}\left[\sum_{k=0}^{n-1}\frac{1}{(2z)^k}\frac{\Gamma(\nu+k+\frac{1}{2})}{k!\Gamma(\nu-k+\frac{1}{2})} + \theta\frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})}\right], \qquad (4.4.7).$$

where $0 \le |\theta| \le 1$, ν and z real and $n \ge \nu - \frac{1}{2}$ (see e.g., Grandshteyn and Ryzhik, eq. 8. 451.6, p. 963).

In view of (4.4.7) and setting $l = \frac{m}{2} - \nu - 1$, we have that

$$S^2_{A,m}(x) = \frac{x}{\sqrt{2}}\,\frac{\sum_{k=0}^{n-1}(2\sqrt{2}x)^{-k}\frac{\Gamma(l+k+\frac{1}{2})}{k!\Gamma(l-k+\frac{1}{2})} + \theta_1\frac{\Gamma(l+n+\frac{1}{2})}{n!\Gamma(l-n+\frac{1}{2})}}{\sum_{k=0}^{n-1}(2\sqrt{2}x)^{-k}\frac{\Gamma(l+k+\frac{3}{2})}{k!\Gamma(l-k+\frac{3}{2})} + \theta_2\frac{\Gamma(l+n+\frac{3}{2})}{n!\Gamma(l-n+\frac{3}{2})}}, \qquad (4.4.8)$$

for $0 \le |\theta_i| \le 1$, and for i=1 or 2.

Thus, as $x \uparrow \infty$, we express the denominator of (4.4.8) as "$1+\alpha$". We then expand this expression in a geometric series, and we keep only the first two terms. After these operations we proceed with the following

$$S^2_{A,m}(x) = \frac{x}{\sqrt{2}}\left\{1 + \frac{1}{2\sqrt{2}x}\frac{\Gamma(l+1+\frac{1}{2})}{\Gamma(l-1+\frac{1}{2})}\left(1 - \frac{l+1+\frac{3}{2}}{l-1+\frac{3}{2}}\right) + o\left(\frac{1}{x}\right)\right\}$$

$$= \frac{x}{\sqrt{2}}\left\{1 - \frac{1}{\sqrt{2}x}(l - \tfrac{3}{2})\right\} + o(1). \qquad 4.4.9)$$

This completes the proof of Theorem 4.1.

4.5 Discussion

In this section we offer graphical presentations of the $\sqrt{E\left[A \mid X_1 = x_1\right]}$ with $x_1 \Sigma^{-1} x_1 = x \geq 0$, for selected values of $l = \frac{m}{2} - \nu - 1$, and for x between 0 and 10. In producing the figures that follow, we have chosen to reduce the parameter space considerably, but in a way that we do not lose the general character of this function. For example, we view $S_{A,m}(x)$ as a function of l and x only, instead of m, ν, and x. From Figure 1, it is clear that as l decreases, $S_{A,m}(x)$ increases, for any fixed value of x, and as x increases, $S_{A,m}(x)$ increases, for any fixed value of l. This is exactly what was expected to be seen. Moreover, from the Theorem we have that for small arguments of x, $S_{A,m}(x) = \sqrt{-(l+1)} + o(1)$, for all l+1<0 This is again in agreement with Figure 2, i.e., $S_{A,m}(x)$ =2+$o(1)$ and $S_{A,m}(x) = 1 + o(1)$, for l=-5 and l=-2, respectively. Also from Figure 2, we observe that, for $l+1 \geq 0$, $S_{A,m}(x)$ starts from the origin, and for l>0, the linear pattern, for small arguments of x, seems to be in order. We know that this is what the Theorem in part II illustrates. Finally, the contour of $S_{A,m}(x)$ answers some of the monotonicity issues we brought up in this Section.

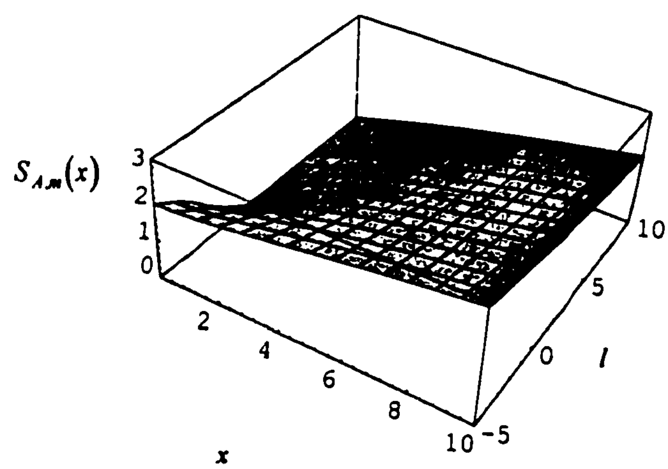

Figure 1: $S_{A,m}(x)$, for $-5 \leq l \leq 10$ and $0 \leq x \leq 10$.

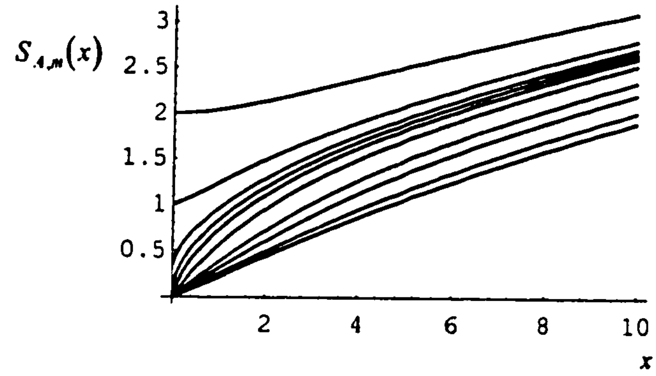

Figure 2: $S_{A,m}(x)$, *for* $0 \le x \le 10$ *and* $l = -5, \ -2, \ -1, \ -.5, \ 0, \ 1, \ 3.2, \ 5, \ 8,$ *and* 10.

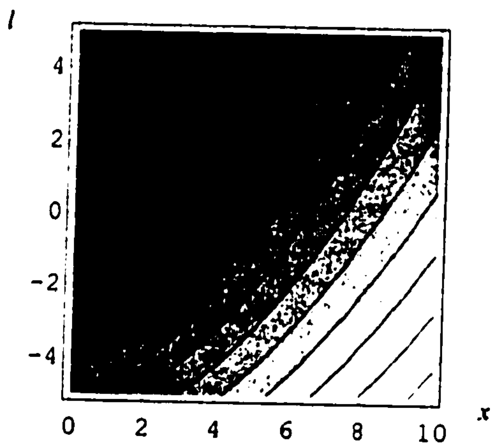

Figure 3: Contour plot of $S_{A,m}(x)$, *for* $0 \le x \le 10$ *and* $-5 \le l \le 5$.

4.6 Concluding Remarks

In this chapter, we provide an exact expression for the conditional variance-covariance matrix. Some results from the special functions enabled us to obtain a simple expression and derive a method of approximating the conditional standard deviation at both small and large arguments. The expression, as well as the approximations, are presented in computable form. We have provided various plots for the non-constant term at selected combinations of the parameters involved. We hope that this theory will answer various questions related to heteroscedastic examples which occur in regression theory and will play a key role in the diagnostic analysis.

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CHAPTER 5

ERROR BOUNDS FOR ASYMPTOTIC EXPANSION OF THE CONDITIONAL VARIANCE OF THE SCALE MIXTURES OF THE MULTIVARIATE NORMAL DISTRIBUTION

5.1 Introduction.

The problem of approximating the scale mixtures of normal distributions has received a lot of interest the last decades. Keilson and Steutel (1974) established moment measures of the distance of mixtures from its parent distribution, and showed that the Pearson's coefficient of kurtosis plays an important role as a metric. Heyde (1975) and Heyde and Leslie (1976) studied the same properties in a greater detail and related the moment measures of distance to more familiar uniform measures. Using a more unified approach, Hall (1979) sharpened Heyde and Leslie's result by reducing a universal constant value. Shimizu (1987, 1995) generalized these results by providing Hermite-type of expansion of these mixtures. In the same framework Fujikoshi and Shimizu (1989) obtained a Hermite-type expansion of multivariate mixture distribution when the scale is distributed in a neighborhood of one in some sense.

This chapter considers the expansion of the conditional variance forms of scale mixture of normal distributions in the same framework as Shimizu (1987). In particular, if $\mathbf{X} \in \mathbf{R}^n, n \geq 2$ is a (non-degenerate) random vector expressed by the stochastic representation $\mathbf{X} = A^{1/2}\mathbf{G}$, where A is a positive random variable independent of the n-dimensional Gaussian random (column) vector $\mathbf{G}$ with mean 0 and positive definite covariance matrix Σ, and the equality is in distribution. In Chapter 3 we have shown that $Cov(\mathbf{X}_2 | \mathbf{X}_1 = \mathbf{x}_1) = E[A | \mathbf{X}_1 = \mathbf{x}_1]\Sigma_{2|1}$, where $\Sigma_{2|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ with $\mathbf{x}_1$ and

$\mathbf{G}_1$ are m-dimensional ($m<n$) and Σ_{11} is $m \times m$-dimensional, *i.e.*, Σ_{11} is the covariance matrix of $\mathbf{G}_1$, etc. It is clear that scale mixtures of normal distributions do not have degenerate conditional variances, as in the normal theory, thus, they provide heteroscedastic examples. Cambanis *et al.* (1997) and Fotopoulos and He (1997) have studied various properties of this conditional variance and obtained several expressions with respect to the moments and/or Laplace transform of A. In this study we investigate the possibility of expanding $E\left[A \middle| \mathbf{X}_1 = \mathbf{x}_1\right]$ in terms of the moments of A and the confluent hypergeometric functions. The expressions are both manageable and in computable form.

Throughout this work, we use vector notation, and $x \wedge 1 = \min(1, x)$ and $x \vee 1 = \max(1, x)$. The organization of this chapter is as follows. The actual expression of the conditional expectation is introduced in Section 5.2. The main results are stated and various comments are suggested. The proofs of the theorems are deferred in Section 5.3. Section 5.4 provides an overview of Laguerre and Hermite polynomials which are connected with the main results. The auxiliary results are displayed in Section 5.5.

5.2 Background and Results

5.2.1 Using Laplace Expressions: In Chapter 3 we have shown that if the Laplace transform of the scale random variable A satisfies

$$\int_{[0,\infty)} u^{m/2-1} E\left[e^{-uA}\right] du < \infty \text{ and } \int_{[0,\infty)} u^{m/2-1} E\left[A e^{-uA}\right] du < \infty \tag{5.2.1}$$

then for $m=1$

$$E\left[A\middle|\mathbf{X}_1=\mathbf{x}_1\right]=\frac{E\left[A\int_0^\infty e^{-t^2 A/2}\cos\left(x_1 t/\sigma_1\right)dt\right]}{E\left[\int_0^\infty e^{-t^2 A/2}\cos\left(x_1 t/\sigma_1\right)dt\right]}, \tag{5.2.2}$$

and for $m>1$

$$E\left[A\middle|\mathbf{X}_1=\mathbf{x}_1\right]=\frac{E\left[A\int_0^\infty t^{m/2}e^{-t^2 A/2}J_{\frac{m-2}{2}}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}t\right)dt\right]}{E\left[\int_0^\infty t^{m/2}e^{-t^2 A/2}J_{\frac{m-2}{2}}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}t\right)dt\right]}. \tag{5.2.3}$$

Evaluating (5.2.2) and/or (5.2.3) can be very difficult. Thus, it is proposed to provide an approximation expression in place of (5.2.2) and (5.2.3), which will, of course, be both manageable and in a computable form.

It is clear that $f(A)=e^{-t^2 A/2}$ has absolutely continuous derivatives of any order on any finite segment $[a,b]\subset(0,\infty)$. Based on this information and the assumption that $\frac{A}{E[A]}$ is close to one, (clarification of the closeness to one will be displayed in Theorems 5.3 and 5.4), in some sense, the conditional $E\left[A\middle|\mathbf{X}_1=\mathbf{x}_1\right]$ is approximated as follows.

THEOREM 5.1 *If $m>1$ and if the Laplace transform of the scale random variable A satisfies (5.2.1) and $E\left[\left(\frac{A}{E[A]}\wedge 1\right)^{-\frac{m+1}{4}}\left|\frac{A}{E[A]}\vee 1-1\right|^k\right]<\infty$ for some $k\in\mathbf{N}$, then the following expansion is in order,*

$$E\left[A\middle|\mathbf{X}_1=\mathbf{x}_1\right]=\frac{\exp\left(-\frac{\|\mathbf{x}\|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)\sum_{j=0}^{k-1}\binom{\frac{m}{2}+j}{j}E\left[A\left(\frac{A}{E(A)}-1\right)^j\right]M\left(-j,\frac{m}{2};\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)+\varepsilon_{1k}\left(\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}},E(A)\right)}{\exp\left(-\frac{\|\mathbf{x}\|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)\sum_{j=0}^{k-1}\binom{\frac{m}{2}+j}{j}E\left[\left(\frac{A}{E(A)}-1\right)^j\right]M\left(-j,\frac{m}{2};\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)+\varepsilon_{2k}\left(\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}},E(A)\right)},$$

where

$$\left|\varepsilon_{ik}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}, E[A]\right)\right| \leq \frac{2^{\frac{m-1}{4}}\,\Gamma\left(\frac{3m-2}{4}+k\right)E[A]^{\frac{m-1}{4}}}{k!\,\Gamma\left(\frac{m}{2}\right)\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}^{\frac{m-1}{4}}}\, E\left[A^{2-i}\left(\frac{A}{E[A]}\wedge 1\right)^{-\frac{m+1}{4}}\left|\frac{A}{E[A]}\vee 1-1\right|^{k}\right] \text{ for } i=1 \text{ or } 2.$$

THEOREM 5.2 *If m=1 and if the Laplace transform of the scale random variable A satisfies (5.2.1) and* $E\left[\left(\frac{A}{E[A]}\wedge 1\right)^{-1/2}\left|\frac{A}{E[A]}\vee 1-1\right|^{k}\right]<\infty$ *for some* $k\in\mathbf{N}$, *then the following expansion is in order.*

$$E[A|X_1=x_1]=\frac{\exp\left(-\frac{x_1^2}{2\sigma_1^2E[A]}\right)\sum_{j=0}^{k-1}\frac{E\left[A\left(\frac{A}{E(A)}-1\right)^j\right]}{j!2^j}H_{2j}\left(\frac{x_1}{\sigma_1(2E[A])^{1/2}}\right)+\varepsilon_{1k}\left(\frac{x_1}{\sigma_1},E(A)\right)}{\exp\left(-\frac{x_1^2}{2\sigma_1^2E[A]}\right)\sum_{j=0}^{k-1}\frac{E\left[\left(\frac{A}{E(A)}-1\right)^j\right]}{j!2^j}H_{2j}\left(\frac{x_1}{\sigma_1(2E[A])^{1/2}}\right)+\varepsilon_{2k}\left(\frac{x_1}{\sigma_1},E(A)\right)},$$

where

$$\left|\varepsilon_{ik}\left(\frac{x_1}{\sigma_1},E[A]\right)\right|\leq\frac{\sqrt{\pi}}{(2E[A])^{1/2}}\frac{1}{2^k}\binom{2k}{k}E\left[A^{2-i}\left(\frac{A}{E[A]}\wedge 1\right)^{-\frac{1}{2}}\left|\frac{A}{E[A]}\vee 1-1\right|^{k}\right] \text{ for } i=1 \text{ or } 2.$$

Remark. (i). Note that if $\frac{A}{E[A]}$ becomes close to one, then $E\left[\left(\frac{A}{E[A]}\wedge 1\right)^{-\frac{m+1}{4}}\left|\frac{A}{E[A]}\vee 1-1\right|^{k}\right]$ $(m>1)$ becomes small as $k\in\mathbf{N}$ increases, thus, we may approximate the conditional expectation of the scalar at a given $\mathbf{x}_1$ by

$$E[A|\mathbf{X}_1=\mathbf{x}_1]\cong\frac{\sum_{j=0}^{k-1}\binom{\frac{m}{2}+j}{j}E\left[A\left(\frac{A}{E(A)}-1\right)^j\right]M\left(-j,\frac{m}{2};\frac{\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}^2}{2E[A]}\right)}{\sum_{j=0}^{k-1}\binom{\frac{m}{2}+j}{j}E\left[\left(\frac{A}{E(A)}-1\right)^j\right]M\left(-j,\frac{m}{2};\frac{\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}^2}{2E[A]}\right)} \qquad (5.2.4)$$

Similarly, under the same conditions as above, i.e. $\frac{A}{E[A]}$ is close to one but now we consider the bivariate case (n=2 and m=1), the $E\left[\left(\frac{A}{E[A]} \wedge 1\right)^{-1/2} \left|\frac{A}{E[A]} \vee 1 - 1\right|^{k}\right]$, for the same reasons as above, becomes close to zero as k increases, hence the conditional expectation is now approximated by

$$E\left[A \middle| X_1 = x_1\right] \cong \frac{\sum_{j=0}^{k-1} \frac{E\left[A\left(\frac{A}{E(A)}-1\right)^{j}\right]}{j!\,2^{j}} H_{2j}\left(\frac{x_1}{\sigma_1 (2E[A])^{1/2}}\right)}{\sum_{j=0}^{k-1} \frac{E\left[\left(\frac{A}{E(A)}-1\right)^{j}\right]}{j!\,2^{j}} H_{2j}\left(\frac{x_1}{\sigma_1 (2E[A])^{1/2}}\right)} \tag{5.2.5}$$

Both (5.2.4) and (5.2.5) are in a computable form, and the accuracy as shown in these two formulas depends on how large a manageable $k \in \mathbf{N}$ is considered.

(ii). Observe that $\binom{\frac{m}{2}+j}{j} M\left(-j, \frac{m}{2}; \frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right) = \binom{\frac{m}{2}+j}{j} {}_1F_1\left(-j, \frac{m}{2}; \frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right) = L_j^{\left(\frac{m}{2}-1\right)}\left(\frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)$, and $L_j^{\left(\frac{m}{2}-1\right)}(x) = \sum_{i=0}^{j} \binom{j+\frac{m}{2}-1}{j-i} \frac{(-x)^i}{i!}$ (see e.g. Rainville p. 203, 1960), where ${}_1F_1$ is the confluent hypergeometric function and $L_j^{(a)}(\cdot)$ is the Laguerre's polynomial, thus (5.2.4) can be written in the form

$$E\left[A \middle| \mathbf{X}_1 = \mathbf{x}_1\right] \cong \frac{\sum_{j=0}^{k-1} E\left[A\left(\frac{A}{E(A)}-1\right)^{j}\right] L_j^{\left(\frac{m}{2}-1\right)}\left(\frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)}{\sum_{j=0}^{k-1} E\left[\left(\frac{A}{E(A)}-1\right)^{j}\right] L_j^{\left(\frac{m}{2}-1\right)}\left(\frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)} \tag{5.2.6}$$

for some $k \in \mathbf{N}$.

Based on the knowledge presented in Section 5.4., it is now grasped in what respect the quantity $\frac{A}{E[A]}$ needs to be closed to one. Furthermore, with the background developed in the same section we alternatively furnish a new representation formula for the conditional expectation of A given $\mathbf{X}_1 = \mathbf{x}_1$ for both $m \geq 2$ and in Theorem 5.4 for $m=1$.

THEOREM 5.3 *If $m>1$ and if the Laplace transform of the scale random variable A satisfies (5.2.1), and if the sequences* $\{\bar{a}_n\}_{n=0}^{\infty} = \left\{E\left[A\left(\frac{A}{E(A)} - 1\right)^n\right]\right\}_{n=0}^{\infty}$ *and* $\{a_n\}_{n=0}^{\infty} = \left\{E\left[\left(\frac{A}{E(A)} - 1\right)^n\right]\right\}_{n=0}^{\infty}$ *satisfy*

$$\lambda_0 = \max\left\{0, \ -\limsup_{n\to\infty}\left(2\sqrt{n}\right)^{-1} \max\left\{\log|\bar{a}_n|, \ \log|a_n|\right\}\right\} < \infty,$$

then

$$E\left[A|\mathbf{X}_1 = \mathbf{x}_1\right] = \frac{\sum_{j=0}^{\infty} E\left[A\left(\frac{A}{E(A)} - 1\right)^j\right] L_j^{\left(\frac{m}{2}-1\right)}\left(\frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)}{\sum_{j=0}^{\infty} E\left[\left(\frac{A}{E(A)} - 1\right)^j\right] L_j^{\left(\frac{m}{2}-1\right)}\left(\frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right)},$$

on every compact subset $\Delta(\lambda_0)$ *of* $\frac{|\mathbf{x}_1|^2_{\Sigma_{11}^{-1}}}{2E[A]}$.

THEOREM 5.4 *If $m=1$ and if the Laplace transform of the scale random variable A satisfies (5.2.1), and if the sequences* $\{\bar{a}_n\}_{n=0}^{\infty} = \left\{E\left[A\left(\frac{A}{E(A)} - 1\right)^n\right]\right\}_{n=0}^{\infty}$ *and* $\{a_n\}_{n=0}^{\infty} = \left\{E\left[\left(\frac{A}{E(A)} - 1\right)^n\right]\right\}_{n=0}^{\infty}$ *satisfy*

$$\tau_0 = \max\left\{0, \ -\limsup_{n\to\infty}(2n+1)^{-1/2} \max\left\{\log\left|\left(2n/e\right)^{\frac{n}{2}} a_n\right|, \log\left|\left(2n/e\right)^{\frac{n}{2}} \bar{a}_n\right|\right\}\right\} < \infty,$$

then

$$E\left[A|\mathbf{X}_1 = \mathbf{x}_1\right] = \frac{\sum_{j=0}^{\infty} \frac{E\left[A\left(\frac{A}{E(A)}-1\right)^j\right]}{j!2^j} H_{2j}\left(\frac{x_1}{\sigma_1(2E[A])^{1/2}}\right)}{\sum_{j=0}^{\infty} \frac{E\left[\left(\frac{A}{E(A)}-1\right)^j\right]}{j!2^j} H_{2j}\left(\frac{x_1}{\sigma_1(2E[A])^{1/2}}\right)}$$

on every compact subset $S(\tau_0)$ *of* $\frac{x_1}{\sigma_1(2E[A])^{1/2}}$.

Remark. In view of Proposition 2, the moments $E\left[A\left(\frac{A}{E(A)}-1\right)^j\right]$, and $E\left[\left(\frac{A}{E(A)}-1\right)^j\right]$ for any $j \in \mathbf{N}$ are then uniquely determined by the expansions presented in the numerator and the denominator, respectively, of both Theorems 5.3 and 5.4.

5.2.2 USING MOMENTS EXPRESSION: Cambanis et al. (1997) have also shown that if $m \geq 2$, or if $m = 1$ and $E\left[A^{1/2}\right] < \infty$, then

$$E\left[A \middle| \mathbf{X}_1 = \mathbf{x}_1\right] = E\left[A^{-m_1/2+1} \exp\left(-\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2A}\right)\right] \Big/ E\left[A^{-m_1/2} \exp\left(-\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2A}\right)\right] \tag{5.2.7}$$

We again here are concerned with an asymptotic expansion of this conditional expectation, under the assumption that the unconditional moments of the scale variable exist.. The proposed result is then formulated as follows

THEOREM 5.5 *If m>1, or if* $m = 1$ *and* $E\left[A^{1/2}\right] < \infty$, *and if the scale random variable A satisfies*

$E\left[\frac{A\left(\frac{A}{E[A]} \vee 1\right)^{k-1}}{\left(\frac{A}{E[A]} \wedge 1\right)^{\frac{m}{2}-1}}\left|A - E[A]\right|^k\right] < \infty$ *for some* $k \in \mathbf{N}$, *then the following ratio expansion is in order,*

$E\left[A \middle| \mathbf{X}_1 = \mathbf{x}_1\right] =$

$$\frac{\dfrac{e^{-\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2E[A]}}}{E[A]^{m/2-1}}\left\{1+\left(\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2}\right)^{-\frac{m}{2}+1}\sum_{j=1}^{k-1}\frac{E\left[A\left(A-E[A]\right)^j\right]}{j!}\lambda_{j,\frac{m}{2}}\left(\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}, E[A]\right)\right\}+\varepsilon_{1k}\left(\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}, E[A]\right)}{\dfrac{e^{-\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2E[A]}}}{E[A]^{m/2}}\left\{1+\left(\frac{\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}}{2}\right)^{-\frac{m}{2}+1}\sum_{j=2}^{k-1}\frac{E\left[\left(A-E[A]\right)^j\right]}{j!}\lambda_{j,\frac{m}{2}}\left(\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}, E[A]\right)\right\}+\varepsilon_{1k}\left(\|\mathbf{x}_1\|^2_{\Sigma_{11}^{-1}}, E[A]\right)},$$

where,

$$\lambda_{j,s}(x,a) = \sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^{i+1} i!\, L_{j-i,s}\left(\frac{x}{2a}\right)a^{-i-1},\quad x \in \mathbf{R}^+$$

$$L_{j,s}(y) = \sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}(s)_i\, y^{-i},\ for\ (s)_i = \frac{\Gamma(s+i)}{\Gamma(s)},\quad y \in \mathbf{R}^+ \ and$$

$$\left|\varepsilon_{ik}\left(t, E[A]\right)\right| \le \sum_{s=0}^{k-1}\sum_{r=0}^{k-s}\binom{k-1}{s}s!\binom{k-s}{r}\left(\tfrac{m}{2}\right)_r\left(\tfrac{t^2}{2}\right)^{-\frac{m}{2}-r+1} E\left[\frac{A^{2-i}\left(A \vee E[A]\right)^{r-1}}{\left(A \wedge E[A]\right)^{\frac{m}{2}-1}}\left|A - E[A]\right|^k\right]$$

for i=1or 2.

In obtaining Theorem 5.5, we introduce the polynomials $L_{j,s}(y)$, $y \in \mathbf{R}^+$. These polynomials are Laguarre-type. However, if the discussed polynomial is expressed with respect to Laguarre polynomial (L.P.) then it will be noticed that the corresponding L.P. has an upper index depending on the lower one. It is well known that for the L.P. we insist that the upper index to be independent to the lower one, because many properties which are valid for the independent case fail to be valid for the dependent one.

Note that for sufficiently large $k \in \mathbf{N}$ the conditional expectation presented in Theorem 5.5, when the scale variable is concentrated to its expected value, may be simplified in the same way as for Theorems 5.1 and 5.2.

5.3 Proofs

Proof of Theorem 5.1. By the Taylor expansion formula, for any $A>0$, we have that if A is fixed, then

$$e^{-t^2 A/2} = e^{-t^2 E[A]/2} \sum_{j=0}^{k-1} (-1)^j \frac{t^{2j}(A-E[A])^j}{j!2^j} + e^{-t^2(E[A]+\theta(A-E[A]))/2} \frac{(-1)^k t^{2k}(A-E[A])^k}{k!2^k}$$

$$= G_k(t,A) + \Delta_k(t,A), \text{ for } \theta \in (0,\ 1) \qquad (5.3.1)$$

In view of the Fubini's theorem of successive integration, Lemma 5.3, and the fact that $M(a,b,z) = e^z M(b-a,b,-z)$ (see e.g., Abramowitz and Stegun 1970, eq. 13.1.27), it follows that for $\theta \in (0,\ 1)$,

$$\int_0^\infty t^{m/2} E[G_k(t,A)] J_{\frac{m-2}{2}}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}} t\right) dt = \frac{2^{m/2}\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^{\frac{m-2}{2}}}{E[A]^{m/2}} \sum_{j=0}^{k-1} \binom{\frac{m}{2}+j}{j} E\left[A\left(\frac{A}{E(A)}-1\right)^j\right] M\left(\frac{m}{2}+j, \frac{m}{2}; -\frac{\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2}{2E[A]}\right)$$

$$= \frac{2^{m/2}\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^{\frac{m-2}{2}} e^{-\frac{\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2}{2E[A]}}}{E[A]^{m/2}} \sum_{j=0}^{k-1} \binom{\frac{m}{2}+j}{j} E\left[A\left(\frac{A}{E(A)}-1\right)^j\right] M\left(-j, \frac{m}{2}; \frac{\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2}{2E[A]}\right)$$

$$= \frac{2^{m/2}\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^{\frac{m-2}{2}}}{E[A]^{m/2}} P_{2,k}\left(\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2, \frac{A}{E(A)}\right), \qquad (5.3.2)$$

Similarly, for $\theta \in (0,\ 1)$,

$$\int_0^\infty t^{m/2} E[\Delta_k(t,A)] J_{\frac{m-2}{2}}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}} t\right) dt = \frac{2^{m/2}\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^{\frac{m-2}{2}} e^{-\frac{\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2}{2(E[A]+\theta(A-E[A]))}}}{E[A]^{m/2}} \binom{\frac{m}{2}+k}{k}$$

$$\times E\left[\frac{1}{\left\{1+\theta\left(\frac{A}{E[A]}-1\right)\right\}^{m/2}} \left(\frac{\frac{A}{E(A)}-1}{1+\theta\left(\frac{A}{E[A]}-1\right)}\right)^k M\left(-k, \frac{m}{2}; \frac{\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2}{2(E[A]+\theta(A-E[A]))}\right)\right]$$

$$= \frac{2^{m/2} \|\mathbf{x}\|_{\Sigma_{11}^{-1}}^{\frac{m-2}{2}}}{E[A]^{m/2}} \varepsilon_{2,k}\left(\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2, \frac{A}{E(A)}\right), \tag{5.3.3}$$

Thus, the denominator of (5.2.3) can be easily revealed. Using the same arguments for the numerator expression (5.2.3) is now in order.

To establish the usefulness and powerfulness of Theorem 5.1, we need to understand the negligibility of the quantity $\varepsilon_{ik}(\cdot,\cdot)$, for i=1, 2. First the following two results are required:

$$M\left(-j, \tfrac{m}{2}, z\right) = \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+j\right)} e^{z} z^{-\frac{m}{2}+1} \int_0^{\infty} e^{-t} t^{\frac{m}{2}+j-1} J_{\frac{m}{2}-1}\left(2\sqrt{zt}\right) dt \tag{5.3.4}$$

(see, e.g., Gradshteyn and Ryzhik, 1980, eq.9.211.3) and

$$J_{\nu}(z) \le \frac{\left|\frac{1}{2} z\right|^{\nu} e^{|\mathrm{Im}(z)|}}{\Gamma(\nu+1)}, \quad \text{for } \nu \ge -\tfrac{1}{2}. \tag{5.3.5}$$

Combining (5.3.4) and (5.3.5), it follows that for $\mathrm{Im}(z) = 0$ and z>0

$$\left|M\left(-j, \tfrac{m}{2}, z\right)\right| \le \frac{1}{\Gamma\left(\frac{m}{2}+j\right)} e^{z} z^{-\frac{m}{2}+1} (z)^{\frac{m-2}{4}} \int_0^{\infty} e^{-t} t^{\frac{m}{2}+j-1} t^{\frac{m-2}{4}} dt = \frac{\Gamma\left(\frac{3m-2}{4}+j\right)}{\Gamma\left(\frac{m}{2}+j\right)} e^{z} z^{-\frac{m-1}{4}}. \tag{5.3.6}$$

Thus, substituting z with $\frac{\|\mathbf{x}\|_{\Sigma_{11}^{-1}}^2}{2\left(E[A]+\theta(A-E[A])\right)}$ in (5.3.6), then the two error terms in (5.2.3) can be bounded as follows

$$\left|\varepsilon_{2k}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}, \frac{A}{E[A]}\right)\right| \le$$

$$\binom{\frac{m}{2}+k}{k} \frac{2^{\frac{m-1}{4}} \Gamma\left(\frac{3m-2}{4}+k\right) E[A]^{\frac{m-1}{4}}}{\Gamma\left(\frac{m}{2}+k\right) \|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}^{\frac{m-1}{4}}} E\left[\left\{1+\theta\left(\frac{A}{E[A]}-1\right)\right\}^{-\frac{m+1}{4}} \left|\frac{\frac{A}{E[A]}-1}{1+\theta\left(\frac{A}{E[A]}-1\right)}\right|^{k}\right], \tag{5.3.7}$$

and

$$\left|\varepsilon_{1k}\left(\|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}, \tfrac{A}{E[A]}\right)\right| \leq$$

$$\binom{\frac{m}{2}+k}{k} \frac{2^{\frac{m-1}{4}} \Gamma\left(\frac{3m-2}{4}+k\right) E[A]^{\frac{m-1}{4}}}{\Gamma\left(\frac{m}{2}+k\right) \|\mathbf{x}_1\|_{\Sigma_{11}^{-1}}^{\frac{m-1}{4}}} E\left[A\left\{1+\theta\left(\tfrac{A}{E[A]}-1\right)\right\}^{-\frac{m-1}{4}} \left|\frac{\frac{A}{E[A]}-1}{1+\theta\left(\frac{A}{E[A]}-1\right)}\right|^k\right] \tag{5.3.8}$$

Note that for $\theta \in (0,\ 1)$

$$1+\theta\left(\tfrac{A}{E[A]}-1\right) \geq \begin{cases} \frac{A}{E[A]} & \text{for } \frac{A}{E[A]} \leq 1 \\ 1 & \text{for } \frac{A}{E[A]} > 1. \end{cases} \tag{5.3.9}$$

Elaborating (5.3.9) the proof of Theorem 5.1 is completed.

Proof of Theorem 5.2 From (5.3.1) and eq. 3.952.9 in Gradshteyn and Ryzhik (1980), it follows that for m=1

$$\int_0^\infty E\left[G_k(t, A)\right] \cos\left(\tfrac{x_1}{\sigma_1} t\right) dt = \sum_{j=0}^{k-1} (-1)^j \frac{E\left[(A-E[A])^j\right]}{j!2^j} \int_0^\infty t^{2j} \cos\left(\tfrac{x_1}{\sigma_1} t\right) e^{-t^2 E[A]/2} dt$$

$$= \frac{\sqrt{\pi}}{(2E[A])^{1/2}} \sum_{j=0}^{k-1} \frac{E\left[\left(\frac{A}{E[A]}-1\right)^j\right]}{j!2^j} e^{-\frac{x_1^2}{2\sigma_1^2 E[A]}} H_{2j}\left(\frac{x_1}{\sigma_1 (2E[A])^{1/2}}\right) \tag{5.3.10}$$

where $H_j(\cdot)$ is the Hermite polynomial.

Similarly, it can be seen that

$$\int_0^\infty E\left[\Delta_k(t, A)\right] \cos\left(\tfrac{x_1}{\sigma_1} t\right) dt =$$

$$\frac{\sqrt{\pi}}{(2E[A])^{1/2}} E\left[\left\{\frac{1}{\left\{1+\theta\left(\frac{A}{E[A]}-1\right)\right\}^{1/2}}\left(\frac{\frac{A}{E[A]}-1}{1+\theta\left(\frac{A}{E[A]}-1\right)}\right)^k \Big/ k!2^k\right\} e^{-x_1^2/2\sigma_1^2(E[A]+\theta(A-E[A]))} H_{2k}\left(\frac{x_1}{\left(2\sigma_1^2\left(E[A]+\theta(A-E[A])\right)\right)^{1/2}}\right)\right]$$

$$= \varepsilon_{2k}\left(\tfrac{x_1}{\sigma_1}, \tfrac{A}{E[A]}\right) \qquad (5.3.11)$$

Since $\int_0^\infty t^{2k} e^{-p^2 t^2} dt = \frac{(2k)!}{2^{k+1} k!(2p)^k}\sqrt{\frac{\pi}{p}}$, then $\varepsilon_{2k}\left(\tfrac{x_1}{\sigma_1}, \tfrac{A}{E[A]}\right)$ can be easily bounded as follows

$$\left|\varepsilon_{2k}\left(\tfrac{x_1}{\sigma_1}, A\right)\right| \le \int_0^\infty E\left[\left|\Delta_k(t, A)\right|\right] dt =$$

$$\frac{\sqrt{\pi}}{(2E[A])^{1/2}} \frac{1}{2^k}\binom{2k}{k} E\left[\left\{1+\theta\left(\tfrac{A}{E[A]}-1\right)\right\}^{-1/2} \left|\frac{\frac{A}{E[A]}-1}{1+\theta\left(\frac{A}{E[A]}-1\right)}\right|^k\right] \qquad (5.3.12)$$

Finally, combining relation (5.3.9) the proof of Theorem 2 is now completed.

Proof of Theorem 5.5. Note that if α, β and γ are continuous mappings from $(0,\infty)$ to any real subset and, β and γ are differentiable with respect to α and β respectively of order k, $k \in \mathbf{N}$, then by the Leibnitz' rule for the k-th derivative of product, we have that

$$\frac{d\gamma}{d\alpha} = \frac{d\gamma}{d\beta}\frac{d\beta}{d\alpha}, \text{ and } \frac{d^k\gamma}{d\alpha^k} = \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{d^{k-j}\gamma}{d\beta^{k-j}}\frac{d^j\beta}{d\alpha^j} \text{ for } k \ge 1 \qquad (3.13)$$

Similarly, by the Leibnitz' rule ,it can be also seen that for $n \in \mathbf{N}$ and $a \in \mathbf{R}^+$

$$D^\ell\left(x^{n+a}e^{-x}\right) = \sum_{k=0}^{\ell}\binom{\ell}{k} D^{\ell-k}\left(e^{-x}\right) D^k\left(x^{n+a}\right)$$

$$= e^{-x}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^{\ell-k}(n+a)_k x^{n+a-k} = x^{n+a}e^{-x} L_{\ell,n+a}(x), \text{ for } \ell \in \mathbf{N} \qquad (5.3.14)$$

where D is the differential operator $\frac{d}{dx}$.

Note that $L_{\ell,n+a}(x)$ is a polynomial of order ℓ of x^{-1}. However, if a=0, then

$$D^{\ell}\left(x^{n}e^{-x}\right)=x^{n}e^{-x}\sum_{k=0}^{d}\binom{\ell}{k}(-1)^{\ell-k}(n)_{k}x^{-k}=x^{n}e^{-x}L_{\ell,n}(x),\text{ for }\ell\in\mathbf{N},\quad(5.3.15)$$

where $d=n\min\left(1,\frac{\ell}{n}\right)$, $\ell\in\mathbf{N}$, i.e., if $\ell>n$ then $L_{\ell,n+a}(x)$ is just a polynomial of order n of x^{-1}.

Set $\gamma(a,t)=a^{-m/2}e^{-\frac{t^{2}}{2a}}$ and $\beta(a,t)=\frac{t^{2}}{2a}$. Note that $\frac{m}{2}$ may be written either as $\nu+\frac{1}{2}$ or ν, for $\nu\in\mathbf{N}$. Thus, combining (5.3.14) and (5.3.15), the following result is in order

$$\frac{d^{\ell}}{da^{\ell}}\left(a^{-m/2}e^{-t^{2}/2a}\right)=\left(\frac{t^{2}}{2}\right)^{-m/2+1}\sum_{j=0}^{\ell-1}\binom{\ell-1}{j}\frac{d^{\ell-j}}{d\left(\frac{t^{2}}{2a}\right)}\left(\left(\frac{t^{2}}{2a}\right)^{m/2}e^{-t^{2}/2a}\right)\frac{d^{j}}{da^{j}}\left(a^{-j}\right)$$

$$=\left(\frac{t^{2}}{2}\right)^{-m/2+1}a^{-m/2}e^{-t^{2}/2a}\sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(-1)^{j+1}j!\,L_{\ell-j,\frac{m}{2}}\left(\frac{t^{2}}{2a}\right)a^{-j-1}$$

$$=\left(\frac{t^{2}}{2}\right)^{-m/2+1}a^{-m/2}e^{-t^{2}/2a}\,\lambda_{\ell,\frac{m}{2}}\left(t^{2},a\right)\qquad(5.3.16)$$

By Taylor's expansion series around $E[A]$, it follows that for fixed A>0

$$A^{-m/2}e^{-\frac{t^{2}}{2A}}=\sum_{j=0}^{k-1}\frac{\left(A-E[A]\right)^{j}}{j!}D^{j}\left(E[A]^{-m/2}e^{-t^{2}/2E[A]}\right)+\Delta_{k}\left(A^{*},t\right)$$

$$=S_{k}(A,t)+\Delta_{k}\left(A^{*},t\right)\qquad(5.3.17)$$

where $A^{*}=E[A]+\theta\left(A-E[A]\right)$ and $\theta\in(0,1)$, and $t=\|\mathbf{x}_{1}\|_{\Sigma_{11}^{-1}}$.

In connection with (5.3.16), we now present an explicit form of the $E\left[S_{k}(A,t)\right]$ as follows.

$$E\left[S_k(A,t)\right]=E[A]^{-m/2}e^{-t^2/2E[A]}\left\{1+\left(\frac{t^2}{2}\right)^{-m/2+1}\sum_{j=2}^{k-1}\frac{E\left[(A-E[A])^j\right]}{j!}\lambda_{j,\frac{m}{2}}\left(\tfrac{t^2}{2},E[A]\right)\right\} \quad (5.3.18)$$

For the residual term, we proceed as follows. Observe that $A^*\geq A$ for $A\leq E[A]$, $A^*\geq E[A]$ for $A\geq E[A]$. Thus, $E\left[\Delta_k(A^*,t)\right]$ can be bounded as

$$E\left[\Delta_k(A^*,t)\right]$$

$$\leq\left(\frac{t^2}{2}\right)^{-m/2+1}\sum_{j=0}^{k-1}\sum_{r=0}^{k-j}\binom{k-1}{j}\binom{k-j}{r}\frac{j!\left(\frac{m}{2}\right)_r}{k!}\left(\frac{t^2}{2}\right)^{-r}E\left[\frac{(A\vee E[A])^r}{(A\wedge E[A])^{\frac{m}{2}-1}}|A-E[A]|^k\right] \quad (5.3.19)$$

This completes the proof of Theorem 5.5.

5.4 Laguerre and Hermite polynomials and series

At this Section we borrow a few standard ideas and definitions from the theory of the classical orthogonal polynomials in order to make our results more revealing and easy to be extrapolated. As a standard reference book it is considered the Rusev (1984).

DEFINITIONS. It is known that every system of orthogonal polynomials $\left\{P_n(z)\right\}_{n=0}^{\infty}$ is linearly independent. In particular, for every integer $\nu\geq 0$, $\left\{P_n(z)\right\}_{n=0}^{\infty}$ is basis in the space of all polynomials with degree not greater than ν. This property together with the orthogonality leads to the important statement that every system of orthogonal polynomials is the solution of a linear recurrence equation of the kind

$$\alpha_n y_{n+1}+(z-\beta_n)y_n+\gamma_n y_{n-1}=0 \quad (5.4.1)$$

where α_n, and $\gamma_n \neq 0$ for $n \in \mathbf{N} - \{0\}$.

In other words, for every $z \in \mathbf{C}$ and $n \in \mathbf{N} - \{0\}$, it is required that

$$\alpha_n P_{n+1}(z) + (z - \beta_n) P_n(z) + \gamma_n P_{n-1}(z) = 0 \tag{5.4.2}$$

Now, if $\alpha_n = n+1$, $\beta_n = 2n + \alpha + 1$, $\gamma_n = n + \alpha$, and $\alpha \in \mathbf{R}\text{-}\{-1, -2, \cdots\}$, then $P_n(z) = L_n^{(\alpha)}(z)$, *i.e.*, they are the Laguerre polynomials.

Let $\{P_n(z)\}_{n=0}^{\infty}$ be a system of polynomials orthogonal in the interval $[a,b]$ with respect to the weight function $w(\cdot)$. This system is a solution of the recurrence equation of the kind(4.1). However, it can be shown that the system of functions

$$Q_n(z) = -\int_a^b \frac{w(t) P_n(t)}{t - z} dt, \quad n \in \mathbf{N}, \tag{5.4.3}$$

holomorphic in the open set $\mathbf{C}\text{-}[a,b]$, is also a solution of (5.4.1). The functions $Q_n(z)$, $n \in \mathbf{N}$, are called functions of second kind. In fact, it can be shown that the system $\{Q_n(z)\}_{n=0}^{\infty}$ is a second solution of the equation (5.4.10) in the open set $\mathbf{C}\text{-}[a,b]$, i.e., $\forall z \in \mathbf{C}\text{-}[a,b]$ the systems $\{P_n(z)\}_{n=0}^{\infty}$ and $\{Q_n(z)\}_{n=0}^{\infty}$ are linearly independent.

Therefore, the Laguerre functions of second kind are given by

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{\alpha} \exp(-t) L_n^{(\alpha)}(t)}{t - z} dt, \quad n \in \mathbf{N}, \tag{5.4.4}$$

where $\alpha > -1$, and $z \in \mathbf{C}\text{-}[a,b]$.

ASYMPTOTIC FORMULAS. If $\alpha \in \mathbf{R}-\{-1, -2, \cdots\}$, the asymptotic behavior of the Laguerre polynomials $\left\{L_n^{(\alpha)}(z)\right\}_{n=0}^{\infty}$ on the ray $(0,\infty)$ is given by Fejer's formulas

$$L_n^{(\alpha)}(x) = \pi^{-1/2} \exp(x/2) x^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \left\{\cos\left((2\pi x)^{1/2} - \tfrac{\alpha\pi}{2} - \tfrac{\pi}{4}\right) + \ell_n^{(\alpha)}(x)\right\}, \tag{5.4.5}$$

where $\ell_n^{(\alpha)}(x) = O\left(n^{-1/2}\right)$ on $x \in (\varepsilon,\omega)$, $0<\varepsilon<\omega<\infty$, for sufficiently large n.

If we are interested only in the growth of $L_n^{(\alpha)}(x)$ as a function of n, we can use the following formula

$$L_n^{(\alpha)}(x) = = O\left(n^{\beta}\right),\ \beta = \max\left\{\tfrac{\alpha}{2} - \tfrac{1}{4}, \alpha\right\}, \tag{5.4.6}$$

which is valid uniformly on every interval $[0,\omega]$, $0<\omega<\infty$, provided that $\alpha \neq \{-1, -2, \cdots\}$ and real.

In view of the rate of convergence, we shall present the asymptotic behavior of Laguerre polynomials if n and z (independently) tend to infinity.

First, we define the following. If $0<\lambda<\infty$, $p(\lambda)$ denotes the image of the straight line $\operatorname{Im}(\omega)=\lambda$ under the transformation $z=\omega^2$. This means that $p(\lambda)$ is the curve that can be described by the equality $\operatorname{Re}(-z)^{1/2}=\lambda$, i.e., it is the parabola with focus at the origin and having the real as its axis. Let $\Delta(\lambda) := \mathit{interior}\left\{p(\lambda): \operatorname{Re}(-z)^{1/2}=\lambda\right\}$. If $0<\lambda<\infty$, $\rho = \max\left\{1, 2\lambda^2\right\}$ and $\alpha \in \mathbf{R}-\{-1, -2, \cdots\}$, then $\exists$ a constant $A = A(\lambda,\rho,\alpha)$: $\forall\ n \in \mathbf{N} - \{0\}$ and $z = x+iy \in \Delta^*(\lambda,\rho) := \bar{\Delta}(\lambda) \cap \{z \in \mathbf{C} : |z| \geq \rho\}$ holds, we have the inequality

$$\left|L_n^{(\alpha)}(z)\right| \leq A|z|^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \exp\left(-z-2\lambda\sqrt{n}\right). \tag{5.4.7}$$

CONVERGENCE OF SERIES IN LAGUERRE POLYNOMIALS. It will be seen that with series in Laguerre polynomials

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z), \quad \alpha \in \mathbf{R}-\{-1, -2, \cdots\} \tag{5.4.8}$$

we have to be careful because their regions of convergence are unbounded and this causes some difficulties. For example, by using only the asymptotic formulas (5.4.5) and (5.4.6) on can not prove a statement like Abel's Lemma for power series.

As before, if $0<\lambda<\infty$, by $\Delta(\lambda) = interior\left\{p(\lambda): \mathrm{Re}(-z)^{\frac{1}{2}} = \lambda\right\}$ and by $\Delta^{*}(\lambda)$, its exterior. By definition $\Delta(0) := \varnothing$ and $\Delta(\infty) := \mathbf{C}$, respectively $\Delta^{*}(0) := \mathbf{C}-[0,\infty)$ and $\Delta^{*}(\infty) := \varnothing$. Further, if $\rho > \max\left\{1, 2\lambda^2\right\}$, we define $\Delta(\lambda,\rho) := \Delta(\lambda) \cap \{z \in \mathbf{C} : |z| < \rho\}$

PROPOSITION 5.1. If $\lambda_0 = \max\left\{0, -\limsup_{n\to\infty}\left(2\sqrt{n}\right)^{-1}\log|a_n|\right\}$, then the (5.4.8) is absolutely uniformly convergent on every compact subset of $\Delta(\lambda_0)$ and divergent in $\Delta^{*}(\lambda_0)$.

To see the absolutely convergence of (5.4.8), inequality (5.4.7) is utilized, namely if $\alpha \in \mathbf{R}-\{-1, -2, \cdots\}$ and $-\limsup_{n\to\infty}\left(2\sqrt{n}\right)^{-1}\log|a_n| \geq \lambda_0$, then, $\forall \lambda \in (0, \lambda_0)$ and $\rho > \max\left\{1, 2\lambda^2\right\}$, the series

$$\sum_{n=1}^{\infty} a_n z^{\frac{\alpha}{2}+\frac{1}{4}} \exp(-z) L_n^{(\alpha)}(z) \tag{5.4.9}$$

is absolutely uniformly convergent on the region $\Delta^{*}(\lambda,\rho)$. Indeed, if $0<\tau<\lambda_0-\lambda$, then $|a_n| = O\left(\exp\left(-(2\lambda+\tau)\sqrt{n}\right)\right)$ and (5.4.7) gives that $\left|a_n z^{\frac{\alpha}{2}+\frac{1}{4}} \exp(-z) L_n^{(\alpha)}(z)\right| = O\left(n^{\frac{\alpha}{2}-\frac{1}{4}} \exp\left(-2\lambda\sqrt{n}\right)\right)$, i.e., the series (5.4.9) is majorized in $\Delta^{*}(\lambda,\rho)$ by

$$\sum_{n=1}^{\infty} n^{\frac{\alpha}{2}-\frac{1}{4}} \exp\left(-\tau\sqrt{n}\right) < \infty. \tag{5.4.10}$$

UNIQUENESS OF THE EXPANSIONS. A well known fact is that the orthogonal polynomials expansions have the property (usually called uniqueness) that if $\sum_{n=0}^{\infty} a_n P_n(z) \equiv 0$, then $a_n \equiv 0 \quad \forall n \in \mathbf{N}$. In other words, the coefficients of an orthogonal expansion are uniquely determined by its sum. For example, in the case of a system of orthogonal $\left\{P_n(z)\right\}_{n=0}^{\infty}$ polynomials on a finite interval $[a,b]$ with respect to weight $w(\cdot)$ the coefficients of a series of the kind $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ are given by the equality

$$a_n = \frac{1}{A_n} \int_a^b w(t) P_n(t) f(t) dt, \ \forall n \in \mathbf{N}, \text{ and } A_n = \int_a^b w(t)\left[P_n(t)\right]^2 dt, \tag{5.4.11}$$

provided that $f(z)$ is uniformly convergent in $[a,b]$.

In the case of Laguerre polynomials $\left\{L_n^{(\alpha)}(z)\right\}_{n=0}^{\infty}$ ($\alpha > -1$) the interval is infinite and we must be careful when applying representation (5.4.11). Rusev (1984) have shown that

PROPOSITION 5.2. Let $0 < \lambda < \infty$ and $\alpha > -1$. If the complex function $f(\cdot)$ has a representation

$$f(z) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z), \quad z \in \Delta(\lambda_0),$$

then $f(\cdot)$ is holomorphic in $\Delta(\lambda_0)$ and $\forall n \in \mathbf{N}$ holds the equality

$$a_n = \frac{1}{I_n^{(\alpha)}} \int_0^{\infty} t^{\alpha} \exp(-t) L_n^{(\alpha)}(t) f(t) dt, \ \forall n \in \mathbf{N}, \text{ and } I_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}.$$

In particular, if $f(z) \equiv 0$, then $a_n \equiv 0 \quad \forall n \in \mathbf{N}$.

HERMITE POLYNOMIALS. It can be seen (see e.g., Rusev, 1984) that

$$H_{2n}(z) = (-1)^n 2^{2n} n!\, L_n^{(-\frac{1}{2})}(z^2), \text{ and } H_{2n+1}(z) = (-1)^n 2^{2n+1} n!\, L_n^{(\frac{1}{2})}(z^2),\ n \in \mathbf{N}. \quad (5.4.12)$$

Thus, the statements presented for Laguerre polynomials could be also be referred to Hermite polynomials. For the sake of convenience, we shall illustrate the following.

We define that $S(\tau) := \{z \in \mathbf{C} : |\mathrm{Im}(z)| < \tau\}$. By definition $S(0) := \varnothing$ and $S(\infty) := \mathbf{C}$. Similarly, $S^*(\tau) := \{z \in \mathbf{C} : |\mathrm{Im}(z)| > \tau\}$, if $0 < \tau < \infty$ and $S^*(0) := \mathbf{C} - (-\infty, \infty)$, and $S^*(\infty) := \varnothing$. Then, the following Abel's Lemma is in order.

PROPOSITION 5.3. *a. If* $\tau_0 := \max\left\{0,\ -\limsup_{n\to\infty} (2n+1)^{-\frac{1}{2}} \log\left|\left(\frac{2n}{e}\right)^{\frac{n}{2}} a_n\right|\right\}$, *then the series* $\sum_{n=0}^{\infty} a_n H_n(z)$ *is absolutely uniformly convergent on every compact subset of* $S(\tau_0)$ *and diverges in* $S^*(\tau_0)$. *And*

b. If a complex function $f(\cdot)$ *has in the strip* $S(\tau_0)$ $(0 < \tau_0 \le \infty)$ *a representation by a series of Hermite polynomials, i.e.,* $f(z) = \sum_{n=0}^{\infty} a_n H_n(z)$, *then* $f(\cdot)$ *is holomorphic in* $S(\tau_0)$ *and* $\forall n \in \mathbf{N}$

$$a_n = \frac{1}{I_n} \int_{-\infty}^{\infty} \exp(-t^2) H_n(t) f(t)\, dt, \text{ and } I_n = \sqrt{\pi}\, 2^n n!.$$

In particular, if $f(z) \equiv 0$, *then* $a_n \equiv 0 \quad \forall n \in \mathbf{N}$.

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CHAPTER 6

ASYMPTOTIC PROPERTIES OF SAMPLE MOMENTS AND SOME UNIT ROOT TEST STATISTICS FOR AN AR(1) PROCESS WITH INFINITE-VARIANCE INNOVATIONS

6.1. Introduction

Many time series data in finance and economics often exhibits non-stationary behavior due to unit roots. In practice, such non-stationary series can be converted to a stationary one by taking appropriate differencing. The appropriateness of differencing hence depends on the detection of the presence of unit roots. Since Fuller (1976) and Dickey and Fuller (1979, 1981), a number of methods for detecting unit root have been proposed for various data generating mechanisms, and the asymptotic behaviors of test statistics have been thoroughly studied (see, Evans and Salvin, 1981, 1984, Bhargava, 1986, Solo,1984, Said and Dickey, 1985, Phillips, 1987a, 1987b, Phillips and Perron, 1988, and many others). All the above results are obtained under a condition that the innovations are either *iid* $(0, \sigma^2)$ or allowed to have certain degree of dependency. The existence of the second moment is a crucial underlying assumption for results obtained by the above authors. In such case, the innovations are in the domain of attraction of a centered Gaussian law, hence weak invariance principle applies to the partial sums according to the functional limit theorem and the limiting distributions can be derived in terms of Wiener process.

In recent years, however, more attention has been given to the possibility that certain phenomena (e.g., stock return data, exchange rate, insurance claims) can be better modeled by distributions with heavier tails than normal distribution. Empirical evidence of heavier tails for the speculative data has been well documented (see, Fama, 1965, Mandelbrot, 1967, and DuMouchel, 1983, etc.). Such observation naturally leads to the consideration of distributions with infinite variance. Moreover, many time series in finance and economics appear to exhibit "discontinuities" (*i.e.*, large jumps) and, thus may be more adequately modeled by time series models whose increments have infinite variance. Perhaps for this reason, recently there has been an increasing interest in modeling financial and economic data with stable process of exponent α, $0 < \alpha < 2$. Like normal distribution for finite-variance case, α-stable distribution is used to model the marginal distribution of infinite-variance data, and the α-stable process and the Levy motion in finite-variance case are the analogs to Gaussian process and Brownian motion in finite case. The major difference is that for the α-stable process, the sample paths are no longer continuous, if $\alpha \in (0, 2)$. Resnick and Greenwood (1979) and Resnick (1986) established the weak invariance principle for appropriate normalized partial sums of a sequence of *iid* random variables from the domain of attraction of a stable law and the sequence of squared r.v.'s. Based on this result, Chan and Tran (1989) obtained the limiting distributions of the *OLS* unit root test statistics for an AR model with noises belonging to the domain of attraction of an α-stable law with $\alpha \in (0, 2)$. Within the same framework, Chan (1990, 1993) obtained the asymptotic results for near-integrated time series, and for the *MA* unit root test statistics for a non-invertible moving average process. Philips (1990) extended the results of Chan and Tran (1989) to allow some moderate degree of dependence and heterogeneity for innovations. Caner (1997) generalized the univariate results to vector autoregressive process. Using Whittle estimator, Mikosch et al. (1995) estimated the *AR* and *MA* coefficients in an *ARMA* process with heavier tailed innovations. Knight (1991) derived the limiting distributions of *M*-estimates of *AR* coefficients for an integrated linear process with infinite-variance innovations, and showed that *M*-estimates have faster rate of convergence than the *LSE* and their asymptotic distributions are conditionally normal or mixed normal, so Wald tests and t-ratios can be constructed. Some other related papers include those by Davis and Resnick (1985a, 1985b, 1986) and Avran and Taqqu (1992), where the limiting theory for sample correlation functions was derived.

Our first goal in this chapter is to develop the asymptotic distribution theory for the unit root test statistics based on the Lagrange Multiplier (*LM*) principle for an integrated autoregressive process with innovations in the domain of attraction of an α -stable law where $\alpha \in (0, 2)$. The *LM* unit root test for finite-variance case was proposed by Schmidt and Phillips (1992) in an attempt of circumventing the difficulty that the distributions of conventional *DF* tests under the null hypothesis are dependent of the nuisance parameters. In order to construct the efficient scores and Hanssenian matrix, one needs to know the explicit form of the likelihood function. Unfortunately, the functional form of density for stable random variable is unknown except for a few cases. So we adopt the *LM* statistic given in Schmidt and Phillips (1992) and assume innovations are heavy-tailed, then derive the limit distribution of *LM* statistic for the infinite variance case. Our second goal is set to derive the asymptotic properties of unit root test statistics based on generalized Durbin-Watson (*DW*) statistics for an *AR* process with heavier-tailed innovations. Since Dickey and Fuller (1981) suggested the use of the DW statistics for the tests of unit root, some work has been done for the *DW*-type unit root tests for the finite-variance case. Sargan and Bhargava (1983) and Bhargava (1986) provided methods using *OLS* residuals in a regression model with drift and time trend and in differenced equation to obtain the *DW*-type test statistics. The exact finite distributions and powers of *DW*-type statistics were also computed using Imhof (1961) routine. Nabeya and Tanaka (1990) suggested a method for the accurate computation of the limiting power under a sequence of local alternatives in the regular *AR* unit root tests. The advantages of the *DW*-type statistics against the *DF*-type statistics for testing the unit root tests may be that the former is easier to calculate the exact finite and limiting distributions, and can be readily extended to the general models and a wide class of tests. In addition, the *DW*-type tests display better power properties in finite samples, especially when the model includes an intercept and/or a linear time trend. In this chapter, we would like to extend the *DW*-type test for unit roots to the infinite-variance case, the limiting properties of *DW* statistics are provided in this chapter. Observing that the parametric unit root tests are too restrictive in many cases, the rank counterpart of Dicky-Fuller unit root test was proposed in Breitung and Gourieroux (1997). Ranked tests are invariant with respect to monotonic transformation and robust against a wide class of outlying observations, and they are expected to perform better than parametric tests. Breitung and Gourieroux (1997) considered asymptotic behaviors of rank test when the innovations are strong white noise series symmetrically distributed around zero. In this chapter, we will extend the results obtained in Breitung and Gourieroux (1997) to the case when the innovations are in the domain of attraction of a symmetric stable law. In an influential paper, Granger and Newbold (1974) examined the likely empirical consequences of nonsense or spurious regressions in econometrics. They argued that the levels of many economic time series are non-stationary and their sample paths are well represented by integrated or near integrated process, and regression equations which relate such time series are often misleading. Phillips (1986) provided an analytical study of regressions involving the levels of economic time series. In his paper, an asymptotic theory was developed for the regression coefficients and for conventional significance tests when regress y_t on x_t while y_t and x_t are two independent random walks with finite variances. In this chapter, we would like to examine the phenomenon of the spurious regression when y_t and x_t are two independent random walks with infinite variances. Large sample asymptotics for regression diagnostic statistics are studied for the case of spurious regression involving two independent random walks with infinite variances.

Chapter 6 is organized as follows. Section 6.2 provides some preliminaries related to *Levy* process and limit distributions, expressed as stochastic integrals of *Levy* motions, of sample moments from a $AR(1)$ model with the innovations belonging to the domain of attraction of a symmetric stable law. The exact densities of these integrals are further studied in this section. Section 6.3 deals with the asymptotic theory for the *LM*-type statistics. In Section 6.4, we consider the limiting theory for *DW*-type statistics. In Section 6.5, we derive the asymptotic theory for the ranked unit root tests when the innovations have heavy tails. Section 6.6 discusses the asymptotic behaviors of diagnostic statistics for a spurious regression in the infinite variance case. Finally, some concluding remarks are provided in Section 6.7.

6.2 Preliminaries

The most common and convenient way to introduce α-stable random variable is to define its the characteristic function (*c.f.*). A random variable X is said to follow the stable distribution if its characteristic function is of the form

$$\phi(t) = \exp\left\{\sigma^{\alpha}\left(|t|^{\alpha} + it\omega(t,\alpha,\beta) + i\mu t\right)\right\},$$

where

$$\omega(t,\alpha,\beta) = \begin{cases} \beta|t|^{\alpha-1}\tan(\pi\alpha/2), & \text{if } \alpha \neq 1, \\ -\beta\dfrac{2}{\pi}\ln|t|, & \text{if } \alpha = 1. \end{cases}$$

where $\alpha \in (0,\ 2]$ is the characteristic exponent characterizing the thickness of tails, $\sigma \geq 0$ is the scale parameter, $\beta \in [-1,\ 1]$ measures skewness of the distribution, and $\mu \in (-\infty,\ \infty)$ is the location parameter. A stable distribution with parameters α, σ, β and μ is denoted by $S_{\alpha}(\sigma,\beta,\mu)$. X is a symmetric a stable ($S\alpha S$)random variable if and only if $\beta = \mu = 0$, and is denoted by $X \sim S_{a}(\sigma,0,0)$.

Let $\{X_n\}$ be a sequence of *iid* random variables with the common distribution *F*. The distribution *F* is said to belong to the domain of attraction of a $S\alpha S$ law if there is a sequence of positive constant $\{a_n\}$ such that

$$\frac{X_1+\cdots+X_n}{a_n} \Rightarrow X \sim S\alpha S, \text{ as } n \to \infty .$$

It is known that the necessary and sufficient condition for *F* to be in the domain of attraction of a stable law is that there is a slowly varying function $l(x)$ such that

$$1 - F(x) \sim px^{-\alpha} l(x), \text{ as } x \to \infty ,$$

where

$$p = \lim_{x\to\infty} [1 - F(x)]/[1 - F(x) + F(-x)].$$

Note that when *F* belongs to the domain of attraction of a symmetric stable law, we have $p = 1/2$.

It is shown (LePage, *et al.*, 1997) that the scaling factor a_n is chosen as

$$a_n = \inf\{x: P(|X| > x) \le n^{-1}\}, \tag{6.2.1}$$

and must satisfy $\lim_{n\to\infty} nP(|X| > a_n x) = x^{-\alpha}$. In general, $a_n = n^{1/\alpha} l_0(n)$, where $l_0(\cdot)$ is a function slowly varying at infinity. Throughout this paper, a_n is a sequence of positive number defined as (6.2.1).

A process $\{X(t),\ t \ge 0\}$ is said to be a stable process if its finite dimension distribution is jointly stable. A stable process $\{L_\alpha(t),\ t \ge 0\}$ is said to be an α stable *Levy* motion if

(1). $L_\alpha(0) = 0$ *a.s.*,

(2). $\{L_\alpha(t),\ t \geq 0\}$ has independent stationary increments, and

(3). $L_\alpha(t) \sim S_\alpha(\sigma t^{1/\alpha}, \beta, 0)$ for any fixed t ($t \geq 0$).

If $\beta = 0$, $\{L_\alpha(t),\ t \geq 0\}$ is called a $S\alpha S$ *Levy* motion; if $\sigma = 1$, it is called a standard *Levy* motion; and if both $\beta = 0$ and $\sigma = 1$, it is called the standard $S\alpha S$ *Levy* motion. Note that for a standard $S\alpha S$ *Levy* motion, $L_\alpha(t) \equiv t^{1/\alpha} L_\alpha(1)$, and for $\alpha = 2$, $L_2(t) \equiv \sqrt{2}W(t)$, where $W(t)$ is the standard Brownian motion since $S_2(\sigma,0,0) \equiv N(0,2\sigma^2)$. The following lemma states the LePage series representation of a standard *Levy* motion, and plays an important role in our analysis.

Lemma 6.1 *A standard $S\alpha S$ Levy process $L_\alpha(t)$ can represented as*

$$L_\alpha(t) =_d (C_\alpha)^{1/\alpha} \sum_{i=1}^{\infty} \delta_i \Gamma_i^{-1/\alpha} 1_{[U_i,1]}(t) \ \textit{for}\ 0 \leq t \leq 1,$$

where $\{\delta_i\}$ is a sequence of iid Radamacher variables satisfying $P(\delta_i = 1) = P(\delta_i = -1) = 1/2$. *$\{U_i\}$ is a sequence of iid uniform random variable over $[0,1]$, and $\{\Gamma_i\}$ is a sequence of arrival times of a unity rate Poisson process, and these three sequences are mutually independent. The constant C_α is a constant defined by*

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1} = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ 2/\pi, & \text{if } \alpha = 1. \end{cases} \tag{6.2.2}$$

This series representation allows some intuitive interpretation for the standard $S\alpha S$ *Levy* motion. In fact, a standard $S\alpha S$ *Levy* motion is a pure jump process, the instants U_i's of the jumps are uniformly distributed over $[0,1]$. It jumps up and down with equal probability. The height of each jump is distributed as the $-1/\alpha$ power of arrival times of a Poisson process with unity arrival rate. The following lemma can be found in Samorodnitsky and Taqqu (1994).

Lemma 6.2. *A random variable* $X \sim S_\alpha\left(C_\alpha^{-1/\alpha},1,0\right)$ *for* $0<\alpha<1$ *has the following series representation*

$$X =_d \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha},$$

where $\{\Gamma_i\}$ *is the same as in Lemma 6.1.*

The weak invariance principle of iid random variables from the domain of attraction of stable law is well known. Recently, LePage, *et al.* (1997) established strong invariance principle for *iid* random variables from the domain of attraction of stable law (not necessary symmetric). This result is sated as the following lemma

Lemma 6.3 *Let* $\{X_i\}$ *be a sequence of iid random variables from the domain of attraction of a* $S_\alpha(\sigma,0,0)$ *law with* $0<\alpha<2$, *then, as* $n \to \infty$, *we have*

(i). $\frac{1}{a_n}\sum_{i=1}^{n} X_i \to \sigma\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha}$, *a.s.*,

(ii). $\frac{1}{a_n}\sum_{i=1}^{[nt]} X_i \to \sigma\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha} 1_{[U_i,1]}(t)$, for $t\in[0,1]$, *a.s., and*

(iii). $\frac{1}{a_n}\sum_{i=1}^{[nt]} |X_i|^r \to \sigma\sum_{i=1}^{\infty}\Gamma_i^{-r/\alpha} 1_{[U_i,1]}(t)$, *a.s. for* $t\in[0,1]$ *and* $r>\alpha$,

where $\{\delta_i\}$, $\{U_i\}$ *and* $\{\Gamma_i\}$ *are defined in Lemma 6.1.*

Note that the convergence is defined on the functional space $D[0,1]$, the set of *cadlag* functions, with Skorohod metric ρ_D defined as

$$\rho_D(x,y) = \inf_{\lambda \in \Lambda}\left[\sup_t \left|x(t) - y(\lambda(t))\right| + \sup_t \left|t - \lambda(t)\right|\right],$$

for all $x, y \in D[0,1]$, where Λ is the set of all continuous increasing real functions $\lambda(t)$ on $[0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$ (see, Gikhman and Skorokhod, 1969).

Lemma 6.4 (Resnick, 1986) *Let* $\{X_t\}$ *be a sequence of iid random variables from the domain of attraction of a* $S_\alpha(\sigma,0,0)$ *law with* $0 < \alpha < 2$, *then, as* $n \to \infty$

$$\left(a_n^{-1}\sum_{t=1}^{\lfloor nt \rfloor} X_t, a_n^{-2}\sum_{t=1}^{\lfloor nt \rfloor} X_t^2\right) \Rightarrow \left(U(t),\ V(t)\right),$$

where $\left(U(t),\ V(t)\right)$ *is a Levy process in* $D[0,1]^2$.

Remark. Lemma 6.4, together with the continuous mapping theorem, is used frequently in the literature to derive the asymptotic properties of test statistics of unit roots. It is worthwhile to get a better understanding of $U(t)$ and $V(t)$. Under the assumption that X_t's belong to the domain of attraction of a symmetric stable law, we can see that, by Lemma 6.1 and Lemma 6.3(ii),

$$U(t) \equiv \sigma C_\alpha^{-1/\alpha} L_\alpha(t) \sim S_\alpha\left(\sigma C_\alpha^{-1/\alpha} t^{1/\alpha},\ 0,\ 0\right),$$

and

$$U(1) \equiv \sigma C_\alpha^{-1/\alpha} L_\alpha(1) \sim \sigma C_\alpha^{-1/\alpha} S_\alpha(1,\ 0,\ 0) =_d \sigma C_\alpha^{-1/\alpha} A^{1/2} Z,$$

where $A =_d \cos(\pi\alpha/4)^{2/\alpha} S_{\alpha/2}(1,1,0)$ and $Z \sim N(0,1)$. And the process $\{V(t),\ t \geq 0\}$ is an $\alpha/2$ totally skewed *Levy* motion. In particular, $a_n^{-2}\sum_{i=1}^n X_i^2 \to V(1)$, *a.s.*. This variable appears frequently in the asymptotics of unit root test statistics. It is known that (Chan and Tran, 1989) $V(1)$ is non-degenerate random variable for $0<\alpha<2$, and $V(1)=1$ for $\alpha=2$. The distribution of $V(1)$ when $0<\alpha<2$ remains unknown in general. Using Lemma 6.2 and Lemma 6.3(iii), we can see that $V(1)$ $\sim\sigma^2 C_{\alpha/2}^{-2/\alpha} S_{\alpha/2}(1,1,0)$, where $C_{\alpha/2}$ is defined in (6.2.2), and $S_{\alpha/2}(1,1,0)$ is a positive totally skewed to the right $\alpha/2$-stable random variable.

The asymptotic results in Lemma 6.3 and Lemma 6.4 can be extended to the linear process case. Let $\{X_t\}$ be a linear process satisfying $X_t=\sum_{j=0}^{\infty}c_j\varepsilon_{t-j}$ where $\{\varepsilon_t\}$ is a sequence of *iid* random variables from the domain of attraction of a $S\alpha S$ law. Under the condition of $\sum_{j=0}^{\infty}|c_j|^{\delta}<\infty$ for $0<\delta<\min(\alpha,1)$ and $\sum_{j=0}^{\infty}c_j\neq 0$, Davis and Resnick (1985) showed that $X_t=\sum_{j=0}^{\infty}c_j\varepsilon_{t-j}$ converges almost surely, and

$$\left(a_n^{-1}\sum_{t=1}^{n}X_t,\ a_n^{-2}\sum_{t=1}^{n}X_t^2\right)\Rightarrow\left(\left(\sum_{j=0}^{\infty}c_j\right)U(1),\ \left(\sum_{j=0}^{\infty}c_j^2\right)V(1)\right).$$

Set $S_t=\sum_{j=1}^{t}X_j$, $\omega=\sum_{j=0}^{\infty}c_j$, and $d^2=\sum_{j=0}^{\infty}c_j^2$, under the condition that $\sum_{j=0}^{\infty}j|c_j|^{\delta}<\infty$ for some positive $\delta<\min(\alpha,1)$, Phillips (1990) showed that a more revealed results can be obtained using Beveridge-Nelson decomposition to $\{\varepsilon_t\}$:

Lemma 6.5 *Let $\{\varepsilon_t\}$ be a linear process defined above. Under condition of $\sum_{j=0}^{\infty}j|c_j|^{\delta}<\infty$ for some positive $\delta<\min(\alpha,1)$, we have*

(i). $\left(na_n^2\right)^{-1}\sum_{t=1}^{n}S_{t-1}^2\Rightarrow\left(\sigma C_{\alpha}^{-1/\alpha}\right)^2 d^2\int_0^1 L_\alpha(r)^2\,dr,$

(ii). $a_n^{-2}\sum_{t=1}^{n}S_{t-1}\varepsilon_t\Rightarrow\left(\sigma C_{\alpha}^{-1/\alpha}\right)^2\omega^2\int_0^1 L_\alpha^-(r)\,dL_\alpha(r),$

and the joint convergence also holds.

Remark. For the linear process case, functional limit theorems can be delicate. In general, the normalized random element $a_n^{-1}\sum_{t=1}^{[nr]}X_t$ may not converge in the usual Skorohod metric, as pointed out in Avram and Taqqu (1992).

Let $\{\varepsilon_t\}$ be a sequence of *iid* random variables in the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with $\alpha \in (0,2)$. Suppose that $\{u_t\}$ is another sequence of *iid* random variables in the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with $\alpha \in (0,2)$, and is independent of $\{\varepsilon_t\}$. It shown in Davis and Resnick (1986), the product $\{\varepsilon_t u_t\}$ is also in the domain of attraction of a $S_\alpha(\sigma,0,0)$ law. That is, $\varepsilon_t u_t$ satisfies

$$\frac{P(|\varepsilon_t u_t|>sx)}{P(|\varepsilon_t u_t|>s)} \to x^{-\alpha} \text{ as } s\to\infty,\ \forall x>0.$$

The appropriate normalization for partial sum $\sum_{t=1}^{n}\varepsilon_t u_t$ is

$$\tilde{a}_n = \inf\{x:\ P(|\varepsilon_t u_t|>x)\le n^{-1}\} = n^{1/\alpha}\tilde{l}_0(n),$$

and the normalized partial sum converges as

$$\tilde{a}_n^{-1}\sum_{t=1}^{n}\varepsilon_t u_t \Rightarrow C_\alpha^{-2/\alpha}\sigma^2 V_\alpha(1),$$

where $V_\alpha(1)$ is a standard *SαS Levy* motion.

Phillips (1990) showed that if ε_t and u_t are two independent random variables in the *normal* domain of attraction of a symmetric stable law, then $\tilde{a}_n = (n\log n)^{1/\alpha}$. It is shown (Davis and Resnick, 1986) that $\tilde{a}_n/a_n \to \infty$, and it is clear that $\tilde{a}_n/a_n^2 \to 0$ because

$$\frac{\tilde{a}_n}{a_n^2} = \frac{n^{1/\alpha} l_0(n)}{\left(n^{1/\alpha} \tilde{l}_0(n)\right)^2} = n^{-1/\alpha} l^*(n),$$

where $l^*(n) = l_0(n) \big/ \left(\tilde{l}_0(n)\right)^2$ is also a function slowly varying at infinity, hence $n^{-1/\alpha} l^*(n) \to 0$. Thus we have

Lemma 6.6 *Let $\{\varepsilon_t\}$ and $\{u_t\}$ be two independent sequences of iid random variables from the domain of attraction of a symmetric stable law, then*

$$a_n^{-2} \sum_{t=1}^{n} \varepsilon_t u_t \to 0 \ a.s.,$$

where a_n is defined in (6.2.1).

Now consider the following autoregressive model

$$Y_t = \rho Y_t + \varepsilon_t, \qquad (6.2.3)$$

where ε_t's are *iid* random variables with a common distribution F belonging to the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with index $\alpha \in (0,2)$. Under H_0: $\rho = 1$, model (6.2.3) can be written as

$$Y_t = S_t + Y_0,$$

where $S_t = \sum_{t=1}^{n} \varepsilon_t$, and $S_0 = 0$. The initial value Y_0 may be either a fixed constant or a random variable whose distribution is independent of the sample size n. Without loss of generality, we may assume $Y_0 = 0$. Collaborating the lemmas listed above, the following theorem is in order

Theorem 6.1 *Let $\{Y_t\}$ be generated from model (6.2.3). Under H_0: $\rho = 1$, as $n \to \infty$, we have* (i). $(na_n)^{-1}\sum_{t=1}^{n} Y_t \Rightarrow \sigma C_\alpha^{-1\,\alpha} \int_0^1 L_\alpha(r)dr$,

(ii). $(na_n^2)^{-1}\sum_{t=1}^{n} Y_{t-1}^2 \Rightarrow (\sigma C_\alpha^{-1\,\alpha})^2 \int_0^1 L_\alpha(r)^2 dr$,

(iii). $(n^2 a_n)^{-1}\sum_{t=1}^{n} tY_t \Rightarrow \sigma C_\alpha^{-1\,\alpha} \int_0^1 rL_\alpha(r)dr$,

(iv). $a_n^{-2}\sum_{t=1}^{n} Y_{t-1}\varepsilon_t \Rightarrow (\sigma C_\alpha^{-1\,\alpha})^2 \int_0^1 L_\alpha^-(r)dL_\alpha(r)$,

(v). $(na_n)^{-1}\sum_{t=1}^{n} t\varepsilon_t \Rightarrow \sigma C_\alpha^{-1\,\alpha} \int_0^1 rdL_\alpha(r)$,

where $\{L_\alpha(r)\}$ *is a SαS Levy motion on* $[0,1]$, $L_\alpha^-(r)$ *is the left limit of* $L_\alpha(t)$.

Proof. Let $S_{[nr]} = \sum_{t=1}^{[nr]}\varepsilon_t$, where $r \in [0,1]$ and $[\cdot]$ denotes the usual integer part. Define the standardized partial sum of ε_t's as

$$X_n(r) = a_n^{-1} S_{[nr]} = \begin{cases} a_n^{-1} S_{t-1}, & \text{for } (t\text{-}1)/\text{n} \le r < t/n, t = 1, 2, \cdots, n, \\ a_n^{-1} S_n, & \text{for } r = 1. \end{cases}$$

By Lemmas 6.1 and 6.3(ii), we have

$$X_n(r) \Rightarrow \sigma C_\alpha^{-1\,\alpha} L_\alpha(r), \text{ for } r \in [0,1] \tag{6.2.4}$$

From (6.2.4) and the continuous mapping theorem, Theorem 6.1 can be established by rewriting those sample moments in terms of functions of $X_n(r)$.

$$\begin{aligned}(i).\ (na_n)^{-1}\sum_{t=1}^{n} Y_{t-1} &= (na_n)^{-1}\sum_{t=1}^{n} S_{t-1} = (n)^{-1}\sum_{t=1}^{n} \frac{S_{t-1}}{a_n} \\ &= (n)^{-1} n \sum_{t=1}^{n} \int_{\frac{t-1}{n}}^{\frac{t}{n}} X_n(r)dr = \int_0^1 X_n(r)dr \\ &\Rightarrow \sigma C_\alpha^{-1\,\alpha} \int_0^1 L_\alpha(r)dr.\end{aligned}$$

$$(ii).\ \left(na_n^2\right)^{-1}\sum_{t=1}^{n}Y_{t-1}^2=\left(na_n^2\right)^{-1}\sum_{t=1}^{n}S_{t-1}^2=(n)^{-1}\sum_{t=1}^{n}\left(\frac{S_{t-1}}{a_n}\right)^2$$
$$=(n)^{-1}n\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}X_n(r)^2dr=\int_0^1X_n(r)^2dr$$
$$\Rightarrow\left(\sigma C_\alpha^{-1/\alpha}\right)^2\int_0^1L_\alpha(r)^2dr.$$

(iii). First, note that $t-1=n^2\int_{\frac{t-1}{n}}^{\frac{t}{n}}rdr-1/2$, and

$\sum_{t=1}^{n-1}tY_t=\sum_{t=1}^{n}(t-1)Y_{t-1}=\sum_{t=1}^{n}(t-1)S_{t-1}=\sum_{t=1}^{n}n^2\int_{\frac{t-1}{n}}^{\frac{t}{n}}rS_{[nr]}dr-\frac{n}{2}\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}S_{[nr]}dr$. Hence

$$\left(n^2a_n\right)^{-1}\sum_{t=1}^{n-1}tY_t=\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}rX_n(r)dr-\frac{1}{2n}\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}X_n(r)dr$$
$$=\int_0^1rX_n(r)dr-\frac{1}{2n}\int_0^1X_n(r)dr$$
$$\Rightarrow\sigma C_\alpha^{-1/\alpha}\int_0^1rL_\alpha(r)dr.$$

Part (iv) was proved in Chan and Tran (1989).

For part (v), note that $\int_{\frac{t-1}{n}}^{\frac{t}{n}}rdS_{[nr]}=rS_{[nr]}\Big|_{\frac{t-1}{n}}^{\frac{t}{n}}-\int_{\frac{t-1}{n}}^{\frac{t}{n}}S_{[nr]}dr=\left(\frac{t}{n}S_t-\frac{t-1}{n}S_{t-1}\right)-\frac{1}{n}S_{t-1}=\frac{t}{n}\varepsilon_t$. So it is true that $t\varepsilon_t=n\int_{\frac{t-1}{n}}^{\frac{t}{n}}rdS_{[nr]}$, thus the following convergence is in order

$$\left(na_n\right)^{-1}\sum_{t=1}^{n-1}t\varepsilon_t=a_n^{-1}\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}rdS_{[nr]}=\int_0^1rdX_n(r)dr$$
$$\Rightarrow\sigma C_\alpha^{-1/\alpha}\int_0^1rdL_\alpha(r).$$

Thus, we complete the proof of Theorem 6.1.

The above asymptotic results are expressed as stochastic integrals of standard *Levy* motion. Despite their initially unfamiliar appearance, they are actually random variables with known densities.

Let us restrict $\{\varepsilon_t\}$ to be a sequence of *iid* $S_\alpha(1,0,0)$ random variables. In this case $a_n = n^{1/\alpha}$, and we have

$$\begin{aligned}(na_n)^{-1}\sum_{t=1}^{n} Y_{t-1} &= n^{-(1+1/\alpha)}\sum_{t=1}^{n} S_{t-1} = n^{-(1+1/\alpha)}\sum_{t=1}^{n-1}(n-t)\varepsilon_t \\ &= n^{-(1+1/\alpha)} S_\alpha\left(\left(\sum_{t=1}^{n-1}(n-t)^\alpha\right)^{1/\alpha}, 0, 0\right) \\ &= n^{-(1+1/\alpha)}\left(\sum_{t=1}^{n-1}(n-t)^\alpha\right)^{1/\alpha} S_\alpha(1,0,0).\end{aligned} \tag{6.2.5}$$

Using Euler's summation

$$f(1)+f(2)+\ \dots\ +f(n) = \int_1^n f(x)dx + o(1),$$

it is not hard to see that

$$\sum_{t=1}^{n-1}(n-t)^\alpha = \sum_{t=1}^{n-1} t^\alpha \approx \int_1^{n-1} t^\alpha dt = \frac{(n-1)^{1+\alpha}}{1+\alpha}.$$

Hence

$$n^{-(1+1/\alpha)}\left(\sum_{t=1}^{n-1}(n-t)^\alpha\right)^{1/\alpha} \to \left(\frac{1}{1+\alpha}\right)^{1/\alpha}, \text{ as } n \to \infty. \tag{6.2.6}$$

In view of (6.2.5) and (6.2.6), the following convergence follows

$$(na_n)^{-1}\sum_{t=1}^{n} Y_{t-1} \Rightarrow S_\alpha\left(\left(\frac{1}{1+\alpha}\right)^{1/\alpha}, 0, 0\right). \tag{6.2.7}$$

On the other hand, from part (i) of Theorem 6.1

$$(na_n)^{-1}\sum_{t=1}^{n} Y_{t-1} \Rightarrow C_\alpha^{-1/\alpha}\int_0^1 L_\alpha(r)dr. \tag{6.28}$$

Thus, the right-hand sides of (6.27) and (6.28) must be equal in distribution

$$\int_0^1 L_\alpha(r)dr =_d C_\alpha^{1\ \alpha}\ S_\alpha\left(\left(\frac{1}{1+\alpha}\right)^{1\ \alpha},\ 0,\ 0\right).$$

If $\alpha = 2$, we have

$$\int_0^1 L_2(r)dr =_d S_2\left(\sqrt{1/3}, 0, 0\right) =_d N(0, 2/3),$$

which is consistent with the known result $\int_0^1 W(r)dr \sim N(0,\ 1/3)$ (Banerjee and Hendry, 1992) since $L_2(r) =_d \sqrt{2}W(r)$.

Following the same line of the proof in Chan and Tran (1989), we have

$$\begin{aligned} a_n^{-2}\sum_{t=1}^{n} Y_{t-1}\varepsilon_t &\Rightarrow \left(\sigma C_\alpha^{-1\ \alpha}\right)^2 \int_0^1 L_\alpha^-(r)dL_\alpha(r) \\ &=_d \frac{\sigma^2}{2}\left(C_\alpha^{-2\ \alpha} L_\alpha(1)^2 - C_{\alpha\ 2}^{-2\ \alpha} S_{\alpha\ 2}(1,1,0)\right) \end{aligned}$$

Note that

$$L_\alpha(1) \sim S_\alpha(1,0,0) =_d A^{1\ 2}Z,$$

where $A \sim S_{\alpha\ 2}\left((\cos\pi\alpha/4)^{2\ \alpha},\ 1,\ 0\right)$ and $Z \sim N(0,2)$, we can rewrite the above result as

$$\begin{aligned} a_n^{-2}\sum_{t=1}^{n} Y_{t-1}\varepsilon_t &\Rightarrow \left(\sigma C_\alpha^{-1\ \alpha}\right)^2 \int_0^1 L_\alpha^-(r)dL_\alpha(r) \\ &=_d \frac{\sigma^2}{2}\left(C_\alpha^{-2\ \alpha} L_\alpha(1)^2 - C_{\alpha\ 2}^{-2\ \alpha} S_{\alpha\ 2}(1,1,0)\right) \\ &=_d \frac{1}{2} c S_{\alpha\ 2}(1,1,0)\left(\chi^2(1) - d\right), \end{aligned}$$

where $c = 2\sigma^2 C_\alpha^{-2\ \alpha}(\cos\pi\alpha/4)^{2\ \alpha}$, and $d = C_{\alpha\ 2}^{-2\ \alpha} \Big/ \left(2C_\alpha^{-2\ \alpha}(\cos\pi\alpha\ 4)^{2\ \alpha}\right)$.

To derive the density of $\int_0^1 rL_\alpha(r)dr$, let $\{\varepsilon_t\}$ be a sequence of *iid* $S_\alpha(1,0,0)$ random variables. First, note that

$$\left(n^2 n^{1/\alpha}\right)^{-1} \sum_{t=1}^{n} tY_t = n^{-(2+1/\alpha)} \sum_{t=1}^{n} \left(\tfrac{n(n+1)}{2} - \tfrac{t(t-1)}{2}\right)\varepsilon_t$$

$$=_d n^{-(2+1/\alpha)} \left(\sum_{t=1}^{n} \left(\tfrac{n(n+1)}{2} - \tfrac{t(t-1)}{2}\right)^{\alpha}\right)^{1/\alpha} S_\alpha(1,0,0). \tag{6.2.9}$$

Let $u = t/(n+1)$, then, $\sum_{t=1}^{n}\left(1 - \frac{t(t-1)}{n(n+1)}\right)^{\alpha} \approx \int_1^n \left(1 - \frac{t(t-1)}{n(n+1)}\right)^{\alpha} dt = (n+1)^{-1} \int_{\frac{1}{n+1}}^{\frac{n}{n+1}} \left(1 - u\left(u + \frac{u-1}{n}\right)\right)^{\alpha} du,$ and

$$\int_{\frac{1}{n+1}}^{\frac{n}{n+1}} \left(1 - u\left(u + \tfrac{u-1}{n}\right)\right)^{\alpha} du \to \int_0^1 \left(1 - u^2\right)^{\alpha} du = (1/2) B\left(\tfrac{1}{2},\ 1+\alpha\right).$$

Thus

$$n^{-(2+1/\alpha)} \left(\sum_{t=1}^{n} \left(\tfrac{n(n+1)}{2} - \tfrac{t(t-1)}{2}\right)^{\alpha}\right)^{1/\alpha} \approx n^{-(2+1/\alpha)} \left(n(n+1)/2\right)(n+1)^{1/\alpha} \left[(1/2) B\left(\tfrac{1}{2}, 1+\alpha\right)\right]^{1/\alpha}$$

$$\to 2^{-(1+1/\alpha)} \left[B\left(\tfrac{1}{2}, 1+\alpha\right)\right]^{1/\alpha}. \tag{6.2.10}$$

Combining (6.2.9) and (6.2.10) we have

$$\left(n^2 a_n\right)^{-1} \sum_{t=1}^{n} tY_t \Rightarrow 2^{-(1+1/\alpha)} \left[B\left(\tfrac{1}{2}, 1+\alpha\right)\right]^{1/\alpha} S_\alpha(1,0,0),$$

and from Theorem 6.1, we know that $\left(n^2 n^{1/\alpha}\right)^{-1} \sum_{t=1}^{n} tY_t \Rightarrow C_\alpha^{-1/\alpha} \int_0^1 r L_\alpha(r) dr$, hence we have

$$\int_0^1 r L_\alpha(r) dr =_d 2^{-(1+1/\alpha)} \left[B\left(\tfrac{1}{2}, 1+\alpha\right)\right]^{1/\alpha} S_\alpha(1,0,0).$$

If $\alpha = 2$

$$\int_0^1 r L_2(r) dr =_d \sqrt{B\left(\tfrac{1}{2}, 3\right)/8}\, S_2(1,0,0) =_d \sqrt{2/15}\, N(0,2) \equiv N(0, 4/15),$$

which is consistent with the known result $\int_0^1 r W(r) dr \sim N(0, 2/15)$.

Table 6.1. Asymptotic Results for Normalized Sample Moments from $AR(1)$ process

Sample moments	Asymtotics	Distribution $0<\alpha<2$	$\alpha=2$
1. $(a_n)^{-1}\sum_{t=1}^{n}\varepsilon_t$	$\sigma C_a^{-1/a}\int_0^1 dL_a(r)=\left(\sigma C_a^{-1/a}\right)L_a(1)$	$\sigma C_a^{-1/a}S_a(1,0,0)$	$N\left(0,2\sigma^2\right)$
2. $(na_n)^{-1}\sum_{t=1}^{n}Y_t$	$\sigma C_a^{-1/a}\int_0^1 L_a(r)dr$	$\sigma C_a^{-1/a}S_a\left(\left(\frac{1}{1+a}\right)^{1/a},\ 0,\ 0\right)$	$N\left(0,\frac{2}{3}\sigma^2\right)$
3. $(na_n)^{-1}\sum_{t=1}^{n}t\varepsilon_t$	$\sigma C_a^{-1/a}\int_0^1 r dL_a(r)$	$\sigma C_a^{-1/a}S_a\left(\left(\frac{1}{1+a}\right)^{1/a},\ 0,\ 0\right)$	$N\left(0,\frac{2}{3}\sigma^2\right)$
4. $\left(n^2a_n\right)^{-1}\sum_{t=1}^{n}tY_t$	$\sigma C_a^{-1/a}\int_0^1 rL_a(r)dr$	$\sigma C_a^{-1/a}2^{-1-1/a}\left[B\left(\frac{1}{2},1+\alpha\right)\right]^{1/a}S_a(1,0,0)$	$N\left(0,\frac{4}{15}\sigma^2\right)$
5. $a_n^{-2}\sum_{t=1}^{n}Y_{t-1}\varepsilon_t$	$\left(\sigma C_a^{-1/a}\right)^2\int_0^1 L_a^-(r)dL_a(r)$	$\frac{1}{2}cS_{a/2}(1,1,0)\left(\chi^2(1)-d\right)$	$\sigma^2\left(\chi^2(1)-1\right)$
6. $\tilde{a}_n^{-1}\sum_{t=1}^{n}\varepsilon_t u_t$	$\left(\sigma C_a^{-1/a}\right)^2V_a(1)$	$\left(\sigma C_a^{-1/a}\right)^2S_a(1,0,0)$	$N\left(0,2\sigma^2\right)$
7. $a_n^{-2}\sum_{t=1}^{n}Y_{t-1}u_t$	$\left(\sigma C_a^{-1/a}\right)^2\int_0^1 L_a^-(r)dU_a(r)$	$\sigma^2C_a^{-1}S_a\left(\left(\int_0^1\lvert L_a(r)\rvert^a dr\right)^{1/a},0,0\right)$	$N\left(0,\ 2\sigma^2\int_0^1 W(r)^2dr\right)$

In the above table, we list asymptotic results of some sample moments and their distributions. The results for *Gaussian* case ($\alpha = 2$) are also provided and served for testing the correctness of the distributional forms of those functionals of *Levy* motions. The density given in 7 of Table 6.1 is conditional on Y_{t-1} and conditioning is valid since $L_\alpha(r)$ and $U_\alpha(r)$ are independent standard $S\alpha S$ *Levy* motions. Thus, the unconditional form may be presented by

$$\left[\left(na_n^2\right)^{-1}\sum_{t=1}^{n}|Y_t|^\alpha\right]^{-1/\alpha}\left(a_\alpha^2\right)^{-1}\sum_{t=1}^{n}Y_t u_t$$
$$\Rightarrow \sigma C_\alpha^{-1/\alpha}\left[\int_0^1|L_\alpha(r)|^\alpha dr\right]^{-1/\alpha}\int_0^1 L_\alpha^-(r)dU_\alpha(r) \sim \sigma C_\alpha^{-1/\alpha}S_\alpha(1,0,0).$$

In Model (6.2.3), the *OLS* estimator of ρ is given by $\hat\rho = \sum_{t=1}^{n}Y_tY_{t-1}\Big/\sum_{t=1}^{n}Y_t^2$, so the conventional (*DF*-type) unit root test statistics are presented by

$$n(\hat\rho - 1) = a_n^{-2}\sum_{t=1}^{n}Y_t\varepsilon_t\Big/ n^{-1}a_n^{-2}\sum_{t=1}^{n}Y_t^2, \text{ and } \hat\tau = \left(\sum_{t=1}^{n}Y_t^2\right)^{1/2}(\hat\rho - 1)\Big/ s,$$

with $s^2 = n^{-1}\sum_{t=1}^{n}\left(Y_t - \hat\rho Y_{t-1}\right)^2$.

Based on the results obtained in Theorem 6.1, we can derive the limiting distributions of the above *DF* unit root test statistics for the infinite variance $AR(1)$ process. The following theorem states the limiting distributions of those test statistics.

Theorem 6.2 *Under* $H_0:\rho = 1$ *in model (6.2.3), we have*

(i). $n(\hat\rho - 1) \Rightarrow \int_0^1 L_\alpha^-(r)dL_\alpha(r)\Big/\int_0^1 L_\alpha(r)^2 dr$,

(ii). $\hat\rho - 1 \to 0$ *in probability,* (iii). $na_n^{-2}s^2 \to W \sim \sigma^2 C_{\alpha/2}^{-2/\alpha} S_{\alpha/2}(1,1,0)$, *a.s.*,

(iv). $\hat{\tau} \Rightarrow \frac{\sigma C_\alpha^{1/\alpha}}{W^{1/2}} \frac{\int_0^1 L_\alpha^-(r)dL_\alpha(r)}{\left(\int_0^1 L_\alpha(r)^2 dr\right)^{1/2}}$,

where W is a positive totally skewed to the right $\alpha/2$-stable random variable.

Theorem 6.2(i) is given is Chan and Tran (1989). Theorem 6.2(ii) is an immediate consequence of (i). (iii) can be proved using the result in Theorem6. 1(ii), and (iv) can be established by the results in (i) and (iii). If the innovation series is a linear process, the limiting distributions of *DF*-type test statistics were given in Phillips (1990).

6.3 Asymptotic Results for the *LM* Statistic

One of the drawbacks of the conventional Dickey-Fuller tests and their variants for a unit root is that the asymptotic forms of the test statistics depend on the assumptions about nuisance parameters representing level and trend in time series models. The meaning of the nuisance parameters under null hypothesis is different from that under the alternative hypothesis. To overcome this drawback, Schmidt and Phillips (1992) proposed to re-parameterize the first-order autoregressive process by

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 t + X_t \\ X_t &= \rho X_t + \varepsilon_t \end{aligned}, \tag{6.3.1}$$

where the meaning of nuisance parameters, β_0 and β_1, remain the same under both hypotheses. Under an assumption that ε_t are *iid* normal, Schmidt and Phillips (1992) developed score tests based on Lagrange multiplier principle for a unit root, that is $\rho = 1$, and showed that the asymptotic distributions of the test statistics are invariant to the nuisance parameters β_0 and β_1. In this section we assume the ε_t's in model (6.3.1) to be *iid* $S_\alpha(1,0,0)$ instead of *iid* $N(0,1)$, we want to develop the asymptotic distribution the LM statistic along the same line of Schmidt and Phillips (1992).

Recall that in Schmidt and Phillips (1992) the LM statistic is constructed as

$$LM = \frac{\left[\sum_{t=2}^{n}\left(Y_t - Y_{t-1} - \tilde{\beta}_1\right)\tilde{S}_{t-1}\right]^2}{\tilde{\sigma}\sum_{t=2}^{n}\tilde{S}_{t-1}^2}, \tag{6.3.2}$$

where $\tilde{\beta}_1$ and $\tilde{\beta}_0^X$ are the restricted MLE's for β_1 and $\beta_0^X = \beta_0 + X_0$ subject to $\rho = 1$ respectively,

$\tilde{S}_{t-1} = Y_{t-1} - \tilde{\beta}_0^X - \tilde{\beta}_1(t-1)$, $\tilde{\sigma} = n^{-1}\sum_{t=1}^{n}\left(Y_t - Y_{t-1} - \tilde{\beta}_1\right)^2$, $\bar{\varepsilon} = \sum_{t=1}^{n}\varepsilon_t/n$, and

$$\tilde{\beta}_1 = (Y_n - Y_1)/(n-1) = \beta_1 + \bar{\varepsilon},$$

$$\tilde{\beta}_0^X = Y_1 - \tilde{\beta},$$

$$\tilde{S}_{t-1} = \sum_{j=2}^{t}\left(\varepsilon_j - \bar{\varepsilon}\right).$$

The following theorem gives the limit distribution of the Lagrange multiplier statistic in (6.3.2) for the infinite variance case.

Theorem 6.3 *In model (6.3.1), assume that* ε_t*'s are iid* $S_\alpha(1,0,0)$ *random variables, then, as* $n \to \infty$,

$$LM = \frac{\left[\sum_{t=2}^{n}\left(Y_t - Y_{t-1} - \tilde{\beta}_1\right)\tilde{S}_{t-1}\right]^2}{\tilde{\sigma}\sum_{t=2}^{n}\tilde{S}_{t-1}^2} \Rightarrow \frac{1}{4}\frac{C_{\alpha/2}^{-2/\alpha}}{C_\alpha^{-2/\alpha}}\frac{S_{\alpha/2}^2(1,1,0)}{\int_0^1 V_\alpha(r)^2\,dr},$$

where $V_\alpha(r) = L_\alpha(r) - rL_\alpha(1)$ is the standard SαS Levy bridge, which is the solution of the stochastic integral equation $V_\alpha(r) = \int_0^r \frac{V_\alpha(s)}{s-1} dr + \int_0^r dL_\alpha(s)$.

Proof. Using Lemma 6.2.3, it is not hard to see that

$$n^{-1\,\alpha}\widetilde{S}_{[nr]} = n^{-1\,\alpha}\sum_{t=1}^{[nr]}(\varepsilon_t - \bar{\varepsilon}) = n^{-1\,\alpha}S_{[nr]} - ([nr]/n)n^{-1\,\alpha}S_n$$

$$\Rightarrow C_\alpha^{-1\,\alpha}\left[L_\alpha(r) - rL_\alpha(1)\right] = C_\alpha^{-1\,\alpha}V_\alpha(r). \tag{6.3.3}$$

Moreover, using (6.3.3) we obtain that

$$n^{-(1+2\,\alpha)}\sum_{t=1}^{n}\widetilde{S}_{t-1}^2 = \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\left(n^{-1\,\alpha}\widetilde{S}_{[nr]}\right)^2 dr = \int_0^1\left(n^{-1\,\alpha}\widetilde{S}_{[nr]}\right)^2 dr$$

$$\Rightarrow \left(C_\alpha^{-1\,\alpha}\right)^2 \int_0^1 V_\alpha(r)^2 dr. \tag{6.3.4}$$

For the numerator of (6.3.2), we proceed as follows

$$n^{-2\,\alpha}\sum_{t=1}^{n}\left(Y_t - Y_{t-1} - \widetilde{\beta}_1\right)\widetilde{S}_{t-1} = n^{-2\,\alpha}\sum_{t=1}^{n}\left(\beta + \varepsilon_t - (\beta + \bar{\varepsilon})\right)\widetilde{S}_{t-1}$$

$$= -\frac{1}{2}\left\{n^{-2\,\alpha}\sum_{t=1}^{n}\varepsilon_t^2 - n^{-1}\left(\sum_{t=1}^{n}\varepsilon_t \Big/ n^{1\,\alpha}\right)^2\right\}$$

$$\Rightarrow (-1/2)C_{\alpha\,2}^{-2\,\alpha}S_{\alpha\,2}(1,1,0). \tag{6.3.5}$$

In addition, note that

$$n^{1-2\,\alpha}\widetilde{\sigma} = n^{-2\,\alpha}\sum_{t=1}^{n}\left(Y_t - Y_{t-1} - \widetilde{\beta}_1\right)^2 = n^{-2\,\alpha}\sum_{t=1}^{n}(\varepsilon_t - \bar{\varepsilon})^2$$

$$\Rightarrow C_{\alpha\,2}^{-2\,\alpha}S_{\alpha\,2}(1,1,0). \tag{6.3.6}$$

Combining (6.3.4), (6.4.5) and (6.3.6), we have

$$LM = \frac{\left[\sum_{t=2}^{n}\left(Y_t - Y_{t-1} - \tilde{\beta}_1\right)\tilde{S}_{t-1}\right]^2}{\tilde{\sigma}\sum_{t=2}^{n}\tilde{S}_{t-1}^2} = \frac{\left[n^{-2/\alpha}\sum_{t=2}^{n}\left(Y_t - Y_{t-1} - \tilde{\beta}_1\right)\tilde{S}_{t-1}\right]^2}{\left(n^{1-2/\alpha}\tilde{\sigma}\right)\left(n^{-(1+2/\alpha)}\sum_{t=2}^{n}\tilde{S}_{t-1}^2\right)}$$

$$= \frac{(-1/2)^2 n^{-2/\alpha}\sum_{t=2}^{n}(\varepsilon_t - \bar{\varepsilon})^2}{n^{-(1+2/\alpha)}\sum_{t=2}^{n}\tilde{S}_{t-1}^2} \Rightarrow \frac{(1/4)C_{\alpha/2}^{-2/\alpha} S_{\alpha/2}(1,1,0)}{C_{\alpha}^{-2/\alpha}\int_0^1 V_\alpha(r)^2\,dr}.$$

This completes the proof of Theorem 6.3.

Remark. (i) The asymptotic result of the *LM* remains the same if ε_t 's are *iid* random variables from the domain of attraction of a *SaS* law. Of course in this case $n^{1/\alpha}$ must be changed to a_n defined in Section 6.2. (ii). If ε_t 's are scale mixture of normal (not independent any more), that is, $\varepsilon_t \sim A^{1/2}Z_t$, where A is some positive random variable, then the asymptotic distribution of the *LM* would be the same as that for ε_t 's being *iid* normal since the *LM* statistics is scale invariant.

6.4 Asymptotic Distributions of Durbin-Watson Statistics

The Durbin-Watson (*DW*) statistics were originally designed to detect the presence of serial correlation of the errors in the regression models. It is known that the *DW* test has good power and certain optimal properties in this case. Dickey and Fuller (1981) suggested the use of the *DW* statistics for the tests of unit root. Saran and Bhargava (1983), Bhargava (1986), Nabeya and Tanaka (1990) developed the *DW*-type test statistics for the unit root tests. Kim (1997) provided the asymptotic percentiles of the *DW* tests for both regular and seasonal cases, and the power of the *DW* using Imhof routine. It was shown numerically (Kim, 1997) that the *DW*-type test statistics have better behaviors against the Dickey-Fuller type test statistics. However, all the above mentioned results were obtained based on the finite variance assumption. In this section, we try to develop the asymptotic distributions of the *DW*-type statistics for the unit root tests based on infinite variance time series models.

Consider the following model

$$Y_t = \mathbf{X}_t \beta + u_t . \tag{6.4.1}$$

If $\{Y_t\}$ and $\{\mathbf{X}_t\}$ are nonstationary time series and $\{u_t\}$ is stationary, then we say $\{Y_t\}$ and $\{\mathbf{X}_t\}$ are cointegrated. But if $\{u_t\}$ is nonstationary, model (6.4.1) is misspecified.

Suppose $\{u_t\}$ in model (6.4.1) satisfies

$$u_t = \phi u_{t-1} + \varepsilon_t , \tag{6.4.2}$$

where $\{\varepsilon_t\}$ is a stationary process. Then $\{u_t\}$ is nonstationary if $\phi = 1$. Note that model (6.4.1) and (6.4.2) are jointly represented by

$$Y_t = \phi Y_{t-1} + \left(\mathbf{x}_t - \phi \mathbf{x}_{t-1}\right)' \beta + \varepsilon_t .$$

The generalized *DW* statistics for testing $H_0 : \phi = 1$ is given by

$$d_k = \frac{\sum_{t=k+1}^{n} \left(\hat{u}_t - \hat{u}_{t-k}\right)^2}{\sum_{t=1}^{n} \hat{u}_t^2}, \quad k = 1, \ldots, n-1 ,$$

where $\hat{u}_t$ are the residuals of the regression model (6.4.1).

6.4.1 Regular Unit Root Test. Now let us consider the regular unit root test for time series model with zero mean. Let $\{Y_t\}$ satisfy the following model

$$Y_t = u_t, \quad u_t = \phi u_{t-1} + \varepsilon_t, \tag{6.4.3}$$

where ε_t are *iid* random variables from the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with index $0<\alpha<2$. For the test of $H_0:\phi=1$, against $H_1:|\phi|<1$, the *DW* type test statistic (see, Tanaka, 1996) is proposed as

$$DW_1 = \frac{\sum_{t=1}^{n}(Y_t - Y_{t-1})^2}{\sum_{t=1}^{n} Y_t^2}.$$

Without loss of generality we assume that $Y_0 = 0$. Under $H_0:\phi = 1$ and by Theorem 6.1, we can see that

$$a_n^{-2}\sum_{t=1}^{n}(Y_t - Y_{t-1})^2 = a_n^{-2}\sum_{t=1}^{n}\varepsilon_t^2 \Rightarrow \sigma^2 C_{\frac{\alpha}{2}}^{-\frac{2}{\alpha}} S_{\frac{\alpha}{2}}(1,1,0), \text{ and}$$

$$\left(na_n^2\right)^{-1}\sum_{t=1}^{n} Y_t^2 \Rightarrow \sigma^2 C_\alpha^{-\frac{2}{\alpha}} \int_0^1 L_\alpha(r)^2 dr. \tag{6.4.4}$$

Now consider the following nonzero mean *AR*(1) model

$$Y_t = \mu + u_t, \quad u_t = \phi u_{t-1} + \varepsilon_t, \tag{6.4.5}$$

where the errors are the same as in model (6.4.3). Note that (6.4.5) can be jointly written as

$$Y_t = (1-\phi)\mu + \phi Y_{t-1} + \varepsilon_t.$$

For the same hypothesis as in (6.4.3), the proposed *DW* test statistic for model (6.4.4) is given by

$$DW_2 = \frac{\sum_{t=2}^{n}(Y_t - Y_{t-1})^2}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2},$$

where $\bar{Y} = n^{-1}\sum_{t=1}^{n} Y_t$.

Under $H_0:\phi = 1$, according to Theorem 6.1,

$$a_n^{-2}\sum_{t=2}^{n}(Y_t - Y_{t-1})^2 = a_n^{-2}\sum_{t=1}^{n}\varepsilon_t^2 - a_n^{-2}\varepsilon_1^2 \Rightarrow \sigma^2 C_{\alpha 2}^{-2\alpha} S_{\alpha 2}(1,1,0), \qquad (6.4.6)$$

and

$$\left(na_n^2\right)^{-1}\sum_{t=1}^{n}(Y_t - \bar{Y})^2 = \left(na_n^2\right)^{-1}\sum_{t=1}^{n}Y_t^2 - \left[\left(na_n\right)^{-1}\sum_{t=1}^{n}Y_t\right]^2$$

$$\Rightarrow \sigma^2 C_\alpha^{-2\alpha}\left\{\int_0^1 L_\alpha(r)^2 dr - \left[\int_0^1 L_\alpha(r)dr\right]^2\right\}$$

$$= \sigma^2 C_\alpha^{-2\alpha}\int_0^1\left\{L_\alpha(r) - \int_0^1 L_\alpha(r)dr\right\}^2 dr. \qquad (6.4.7)$$

Collecting the above results (6.4.4), (6.4.6) and (6.4.7), we have the following theorem:

Theorem 6.4 *I. Let $\{Y_t\}$ satisfy (6.4.3), then under $H_0:\phi = 1$, the limiting distribution of nDW_1 is given by*

$$nDW_1 \Rightarrow \frac{C_{\alpha 2}^{-2\alpha} S_{\alpha 2}(1,1,0)}{C_\alpha^{-2\alpha}\int_0^1 L_\alpha(r)^2 dr};$$

II. If $\{Y_t\}$ is generated by (6.4.5), then under $H_0:\phi = 1$, the limiting distribution of nDW_2 is given by

$$nDW_2 \Rightarrow \frac{C_{\alpha 2}^{-2\alpha} S_{\alpha 2}(1,1,0)}{C_\alpha^{-2\alpha}\int_0^1\left\{L_\alpha(r) - \int_0^1 L_\alpha(r)dr\right\}^2 dr}.$$

Remark. If ε_t's have $\alpha/2$-stable mixtures of normal distributions, *i.e.*, they are radically decomposable, then the limiting distribution under null hypothesis would be the same as it is for the normal innovations case. The exact distribution and the exact power of DW_1 and DW_2 can be obtained using the Imhof routine (Sargan and Bhargava, 1983, Bhargava, 1986). If $\{\varepsilon_t\}$ is a linear process with infinite variance, the following result can be obtained using Lemma 6.2.5.

Corollary 6.4.1. *In model (6.4.3), if $\{\varepsilon_t\}$ is linear process, i.e., $\varepsilon_t = \sum_{j=1}^{\infty} c_j u_{t-j}$ satisfying $\sum_{j=1}^{\infty} j|c_j|^{\delta} < \infty$, where $\delta = 1 \wedge \alpha$. Then, under $H_0: \phi = 1$,*

$$nDW_1 \Rightarrow \frac{\left(C_{\alpha/2}^{-2/\alpha} \sum_{j=0}^{\infty} c_j^2\right) S_{\alpha/2}(1,1,0)}{C_{\alpha}^{-2/\alpha} \left(\sum_0^{\infty} c_j\right)^2 \int_0^1 L_\alpha(r)^2 \, dr}.$$

6.4.2 Seasonal Unit Root Test. The asymptotic results can be extended to seasonal time series models. Let us consider the following zero mean seasonal model

$$Y_t = u_t, \quad u_t = \Phi\, u_{t-s} + \varepsilon_t, \tag{6.4.8}$$

where s is the period of seasons, ε_t's are *iid* random variables from the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with index $0 < \alpha < 2$. For the test of $H_0: \Phi = 1$, against $H_1: |\Phi| < 1$ under model (6.4.8), the proposed DW type test statistic is given by

$$DW_3 = \frac{\sum_{t=1}^{n} (Y_t - Y_{t-s})^2}{\sum_{t=1}^{n} Y_t^2}.$$

Without loss of generality we may assume that $n = ms$ and $Y_{-s+1} = \cdots = Y_0 = 0$. Let $t = (l-1)s + j$ $(l = 1,\ldots,m;\ j = 1,\ldots,s)$, then by Theorem 6.1, as $n \to \infty$, $m \to \infty$, we have

$$a_m^{-1}\sum_{l=1}^{[mr]}\varepsilon_{(l-1)s+j} \Rightarrow \sigma C_\alpha^{-1/\alpha} L_\alpha^{(j)}(r), \tag{6.4.9}$$

and

$$a_m^{-2}\sum_{l=1}^{m}\varepsilon_{(l-1)s+j}^2 \Rightarrow \left(\sigma C_\alpha^{-1/\alpha}\right)^2 W_j, \tag{6.4.10}$$

where $L_\alpha^{(j)}(r)$'s are the mutually independent standard *Levy* motions and $W_j \sim S_{\alpha/2}(1,1,0)$ *iid*, corresponding to the partial sums of ε_t belonging to the j-th season. Recall that $a_n = n^{1/\alpha} l_0(n)$ where $l_0(n)$ is slowly varying at infinity, and $n = ms$ for some positive seasonal period s, we have

$$\frac{a_n}{a_m} = \frac{n^{1/\alpha} l_0(n)}{m^{1/\alpha} l_0(m)} = s^{1/\alpha}\frac{l_0(ms)}{l_0(m)} \to s^{1/\alpha}, \text{ as } m \to \infty, \tag{6.4.11}$$

and

$$\frac{na_n^2}{ma_m^2} = \frac{n^{1+2/\alpha}}{m^{1+2/\alpha}}\frac{l_0(n)}{l_0(m)} = s^{1+2/\alpha}\frac{l_0(ms)}{l_0(m)} \to s^{1+2/\alpha} \text{ as } m \to \infty, \tag{6.4.12}$$

by the definition of slowly varying function.

From (6.4.9) - (6.4.12), we obtain

$$a_n^{-2}\sum_{t=1}^{n}(Y_t - Y_{t-s})^2 = \frac{a_m^2}{a_n^2}\sum_{j=1}^{s}\left(a_m^{-2}\sum_{l=1}^{m}\varepsilon_{(l-1)s+j}^2\right)$$

$$\Rightarrow \left(\sigma^2 C_{\alpha/2}^{-2/\alpha}\right)s^{-2/\alpha}\sum_{j=1}^{s} W_j =_d \left(\sigma^2 C_{\alpha/2}^{-2/\alpha}\right)S_{\alpha/2}(1,1,0), \tag{6.4.13}$$

since W_j's are *iid* $S_{\alpha/2}(1,1,0)$ and hence $\sum_{j=1}^{s} W_j =_d s^{2/\alpha} S_{\alpha/2}(1,1,0)$.

Moreover, by Theorem 6.1,

$$\left(ma_m^2\right)^{-1}\sum_{t=1}^{n}Y_t^2=\sum_{j=1}^{s}\left(\left(ma_m^2\right)^{-1}\sum_{l=1}^{m}Y_{(l-1)s+j}^2\right)$$

$$\Rightarrow\sigma^2C_\alpha^{-2\alpha}\sum_{j=1}^{s}\int_0^1 L_\alpha^{(j)}(r)^2\,dr. \tag{6.4.14}$$

In view of (6.4.12), (6.4.13) and (6.4.14), we have

$$nDW_3=n\frac{\sum_{t=1}^{n}\left(Y_t-Y_{t-s}\right)^2}{\sum_{t=1}^{n}Y_t^2}=n\frac{a_n^2}{ma_m^2}\frac{a_n^{-2}\sum_{t=1}^{n}\left(Y_t-Y_{t-s}\right)^2}{\left(ma_m^2\right)^{-1}\sum_{t=1}^{n}Y_t^2}$$

$$\Rightarrow\frac{C_{\alpha 2}^{-2\alpha}}{C_\alpha^{-2\alpha}}\frac{s^{1+2\alpha}S_{\alpha 2}(1,1,0)}{\sum_{j=1}^{s}\int_0^1 L_\alpha^{(j)}(r)^2\,dr}. \tag{6.4.15}$$

We also consider the asymptotic distribution of the *DW* statistic for a univariate time series with nonzero seasonal mean model:

$$Y_t=\sum_{j=1}^{s}\beta_j\delta_{jt}+u_t,\quad u_t=\Phi u_{t-1}+\varepsilon_t, \tag{6.4.16}$$

where ε_t 's are independent both inter-seasons and intra-seasons and in the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with $0<\alpha<2$, $\delta_{jt}=1$ if $j\equiv t(\text{mod } s)$ or 0 otherwise, $\delta_{st}\equiv\delta_{0t}$, and s is the seasonal period. Note that model (6.4.16) can be jointly written as

$$Y_t=(1-\Phi)\sum_{j=1}^{s}\beta_j\delta_{jt}+\Phi Y_{t-s}+\varepsilon_t.$$

To test $H_0:\Phi=1$, against $H_1:|\Phi|<1$, the following *DW* statistic is proposed (see, Kim, 1997),

$$DW_4=\frac{\sum_{t=s+1}^{n}\left(Y_t-Y_{t-s}\right)^2}{\sum_{t=1}^{n}\left(Y_t-\sum_{j=1}^{s}\bar{Y}_j\delta_{jt}\right)^2},$$

where $\bar{Y}_j = m^{-1}\sum_{l=1}^{m} Y_{(l-1)s+j}$ is the *OLS* estimate of the j-th seasonal mean β_j.

Note that $\sum_{t=s+1}^{n}(Y_t - Y_{t-s})^2 = \sum_{j=1}^{s}\sum_{l=2}^{m}\varepsilon_{(l-1)s+j}^2$, and by (6.4.10)

$$a_m^{-2}\sum_{l=2}^{m}\varepsilon_{(l-1)s+j}^2 = a_m^{-2}\sum_{l=1}^{m}\varepsilon_{(l-1)s+j}^2 - a_m^{-2}\varepsilon_j^2 \Rightarrow \sigma^2 C_{\alpha 2}^{-2\alpha} W_j \text{ for } j = 1, \ldots, s.$$

Hence

$$a_n^{-2}\sum_{t=s+1}^{n}(Y_t - Y_{t-s})^2 = \frac{a_m^2}{a_n^2}\sum_{j=1}^{s}\left(a_m^{-2}\sum_{l=2}^{m}\varepsilon_{(l-1)s+j}^2\right)$$

$$\Rightarrow \left(\sigma^2 C_{\alpha 2}^{-2\alpha}\right)s^{-2\alpha}\sum_{j=1}^{s} W_j =_d \left(\sigma^2 C_{\alpha 2}^{-2\alpha}\right)S_{\alpha 2}(1,1,0). \qquad (6.4.17)$$

Furthermore, observe that $\sum_{t=1}^{n}\left(Y_t - \sum_{j=1}^{s}\bar{Y}_j\delta_{jt}\right)^2 = \sum_{j=1}^{s}\sum_{l=1}^{m}\left(Y_{(l-1)s+j} - \bar{Y}_j\right)^2$, and

$$a_m^{-2}\sum_{l=1}^{m}\left(Y_{(l-1)s+j} - \bar{Y}_j\right)^2 \Rightarrow \sigma^2 C_\alpha^{-2\alpha}\int_0^1\left\{L_\alpha^{(j)}(r) - \int_0^1 L_\alpha^{(j)}(r)dr\right\}^2 dr \text{ for } j = 1, \ldots, s,$$

we have

$$\left(ma_m^2\right)^{-1}\sum_{t=1}^{n} Y_t^2 = \sum_{j=1}^{s}\left(\left(ma_m^2\right)^{-1}\sum_{l=1}^{m} Y_{(l-1)s+j}^2\right)$$

$$\Rightarrow \sigma^2 C_\alpha^{-2\alpha}\sum_{j=1}^{s}\int_0^1\left\{L_\alpha^{(j)}(r) - \int_0^1 L_\alpha^{(j)}(r)dr\right\}^2 dr. \qquad (6.4.18)$$

The results (6.4.17) and (6.4.18), together with (6.4.12) yield

$$nDW_4 = n\frac{\sum_{t=s+1}^{n}(Y_t - Y_{t-s})^2}{\sum_{t=1}^{n}\left(Y_t - \sum_{j=1}^{s}\bar{Y}_j\delta_{jt}\right)^2} = \frac{na_n^2}{ma_m^2}\frac{a_n^{-2}\sum_{t=1}^{n}(Y_t - Y_{t-s})^2}{\sum_{j=1}^{s}\left(ma_m^2\right)^{-1}\sum_{l=1}^{m}\left(Y_{(l-1)s+j} - \bar{Y}_j\right)^2}$$

$$\Rightarrow \frac{C_{\alpha 2}^{-2\alpha}}{C_\alpha^{-2\alpha}}\frac{s^{1+2\alpha}S_{\alpha 2}(1,1,0)}{\sum_{j=1}^{s}\int_0^1\left\{L_\alpha^{(j)}(r) - \int_0^1 L_\alpha^{(j)}(r)dr\right\}^2 dr} \qquad (6.4.19)$$

Collecting result (6.4.15) and (6.4.19), the following theorem is then in order:

Theorem 6.5 *I. Let $\{Y_t\}$ be generated from model (6.4.8) where ε_t's are independent both inter-seasons and intra-seasons and in the domain of attraction of a $S_\alpha(\sigma,0,0)$ law with $0 < \alpha < 2$. The limiting distribution of nDW_3 under H_0: $\Phi = 1$ is given by*

$$nDW_3 \Rightarrow \frac{C_{\alpha/2}^{-2\alpha}}{C_\alpha^{-2\alpha}} \frac{s^{1+2\alpha} S_{\alpha/2}(1,1,0)}{\sum_{j=1}^{s} \int_0^1 L_\alpha^{(j)}(r)^2 \, dr};$$

II. If $\{Y_t\}$ is generated by the nonzero mean seasonal model (6.4.16) with same assumption for ε_t as in model 96.4.4), the limiting distribution of nDW_4 under H_0: $\Phi = 1$ is given by

$$nDW_4 \Rightarrow \frac{C_{\alpha/2}^{-2\alpha}}{C_\alpha^{-2\alpha}} \frac{s^{1+2\alpha} S_{\alpha/2}(1,1,0)}{\sum_{j=1}^{s} \int_0^1 \left\{ L_\alpha^{(j)}(r) - \int_0^1 L_\alpha^{(j)}(r) dr \right\}^2 dr},$$

where $L_\alpha^{(j)}(r)$'s are the mutually independent standard Levy motions corresponding to the partial sums of ε_t belonging to the j-th season.

6.4.3 Simultaneous Tests for Both Regular and Seasonal Unit Roots. In what follows, we consider the simultaneous tests of the both regular and seasonal unit roots for a zero mean time series model:

$$Y_t = u_t, \; (1 - \phi B)(1 - \Phi B^s)u_t = \varepsilon_t, \tag{6.4.20}$$

where ε_t's are *iid* random variables from the domain of attraction of a symmetric stable law with index α, $0 < \alpha < 2$ for both inter-seasons and intra-seasons, and B is the backshift operator. For the simultaneous test of the null hypothesis

$$H_0 : (\phi, \Phi) = (1, 1)$$

verses the alternative hypothesis

$$H_a : -1 < \phi, \Phi \le 1 \text{ and } (\phi, \Phi) \ne (1, 1),$$

the *DW* test statistic is proposed (Kim, 1997) as:

$$DW_S = \frac{\sum_{t=1}^{n}\left((1-B)(1-B^s)Y_t\right)^2}{\sum_{t=1}^{n} Y_t^2} = \frac{\sum_{t=1}^{n}\left(Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}\right)^2}{\sum_{t=1}^{n} Y_t^2}.$$

Without loss of generality, we may assume that $n = ms$ and $Y_{-s} = \cdots = Y_0 = 0$. Under H_0, if we write $(1-B)Y_t = N_t$, then $Y_t = Y_{t-1} + N_t$ and $N_t = N_{t-s} + \varepsilon_t$. Thus,

$$Y_t = \sum_{i=1}^{t} N_i = \sum_{j=1}^{s}\sum_{l=1}^{[(t-j+s)/s]} N_{(l-1)s+j} \text{ and } N_{(l-1)s+j} = \sum_{k=1}^{l}\varepsilon_{(k-1)s+j}.$$

Therefore,

$$(ma_m)^{-1} Y_{[nr]} = (ma_m)^{-1}\sum_{j=1}^{s}\sum_{l=1}^{[mr]} N_{(l-1)s+j} + o_p(1) \Rightarrow \sigma C_\alpha^{-1/\alpha}\sum_{j=1}^{s}\int_0^r L_\alpha^{(j)}(r_1)dr_1 \equiv \sigma C_\alpha^{-1/\alpha} B_Y(r),$$

and hence

$$n^{-1}(ma_m)^{-2}\sum_{t=1}^{n} Y_t^2 = n^{-1}(ma_m)^{-2}\sum_{t=1}^{n} Y_{t-1}^2 + o_p(1)$$

$$= \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\left(Y_{[nr]}/(ma_m)\right)^2 dr + o_p(1)$$

$$= \int_0^1 \left(Y_{[nr]}/(ma_m)\right)^2 dr + o_p(1)$$

$$\Rightarrow \left(\sigma^2 C_\alpha^{-2\alpha}\right)\int_0^1 B_Y(r)^2 dr. \qquad (6.4.21)$$

In addition,

$$a_n^{-2}\sum_{t=1}^{n}\left((1-B)(1-B^s)Y_t\right)^2 = \frac{a_m^2}{a_n^2}\sum_{j=1}^{s} a_m^{-2}\sum_{l=1}^{m}\varepsilon_{(l-1)s+j}^2$$

$$\Rightarrow \left(\sigma^2 C_{\alpha 2}^{-2\alpha}\right) s^{-2\alpha}\sum_{j=1}^{s} W_j =_d \left(\sigma^2 C_{\alpha 2}^{-2\alpha}\right) S_{\alpha 2}(1,1,0) \qquad (6.4.22)$$

Combining (6.4.21) and (6.4.22), we have

$$n^3 DW_5 = \left(\frac{n^3 a_n^2}{nm^2 a_m^2}\right)\frac{a_n^{-2}\sum_{t=1}^{n}\left((1-B)(1-B^s)Y_t\right)^2}{n^{-1}(ma_m)^{-2}\sum_{t=1}^{n} Y_t^2}$$

$$\Rightarrow \frac{C_\alpha^{2\alpha}}{C_{\alpha 2}^{2\alpha}}\frac{s^{2+2\alpha}S_{\alpha 2}(1,1,0)}{\int_0^1 B_Y(r)^2 dr}.$$

The following theorem is then in order:

Theorem 6.6 *Under* $H_0: (\phi, \Phi) = (1, 1)$ *for model (6.4.20), the normalized DW statistic for the simultaneous unit root tests* $n^3 DW_5$ *has the following asymptotic distribution:*

$$n^3 DW_5 \Rightarrow \frac{C_\alpha^{2\alpha}}{C_{\alpha 2}^{2\alpha}}\frac{s^{2+2\alpha}S_{\alpha 2}(1,1,0)}{\int_0^1 B_Y(r)^2 dr},$$

where $B_Y(r) = \sum_{j=1}^{s}\int_0^1 L_\alpha^{(j)}(r_1)dr_1$ *and* $L_\alpha^{(j)}(r)$ *'s are the mutually independent standard SαS Levy motions on* $[0,1]$.

Remark. If ε_t's are independent inter-seasons and but not independent intra-seasons, the limiting distributions of nDW_3 and nDW_4 in Theorem 6.5, and $n^3 DW_5$ in Theorem 6.6 would be different, since $\sum_{j=1}^{s} W_j =_d s^{2/\alpha} S_{\alpha/2}(1,1,0)$ requires that W_j's be *iid.*

6.5 Asymptotics of the Ranked Dickey-Fuller Unit Root Test Statistics

It was argued in Breitung and Gourieroux (1997) that the rank counterpart of the conventional Dickey-Fuller unit root tests is advantageous over the parametric tests. The ranked test reduces the influence of outlying observations, and is unaffected by the choice of the initial transformation applied to time series before the unit root test. In Breitung and Gourieroux (1997), the ranked Dickey-Fuller test statistics were proposed for the test of hypothesis that the series is a monotonic transformation of a ramdom walk. Under the assumption of the existence of the second moment, it was shown that the sequence of ranks built from the levels of time series does not converge to a functional of Brownian motion, the asymptotic properties of the rank test are hence different from its parametric counterpart. In this section, we want to investigate the asymptotic properties of the ranked Dickey-Fuller unit root test for the infinite-variance time series. Let $\{Y_t\}$, $t = 1, \ldots, n$ be a series of observations, and let h be a monotonic function such that $Z_t = h(Y_t)$ satisfies the following $AR(1)$ model

$$Z_t = \rho Z_{t-1} + \varepsilon_t \,, \; t = 1, \; \ldots, \; n \,, \tag{6.5.1}$$

where $\{\varepsilon_t\}$ is a sequence of *iid* random variables from the domain of attraction of $S_\alpha(\sigma,0,0)$ law with index α, $0 < \alpha < 2$. For the null hypothesis that the transformed series $\{Z_t\}$ is generated by a random walk with $\rho = 1$, that is, H_0: $\{\exists h$ monotonic: $h(Y_t) = h(Y_{t-1}) + \varepsilon_t\}$, the following testing procedure is proposed. We firstly constructed a series of ranks $\{r_t\}$

$$r_t = \text{Rank of } h(Y_t) \text{ among } h(Y_0),\ \ldots,\ h(Y_n) - (n+1)/2$$

$$\equiv \text{Rank of } Y_t \text{ among } Y_0,\ \ldots,\ Y_n - (n+1)/2\,. \tag{6.5.2}$$

The second equality holds because h is assumed to be monotonic. Under H_0: $\rho = 1$, model (6.5.1) can be written as

$$Z_t = Z_0 + \sum_{j=1}^{t} \varepsilon_j\,.$$

Let $Z_0 = 0$ for simplicity. As shown in Theorem 6.1, we have

$$a_n^{-1} Z_{[sn]} = a_n^{-1} \sum_{t=1}^{[sn]} \varepsilon_t \Rightarrow \left(C_\alpha^{-1\,\alpha}\right) L_\alpha(s),$$

which follows that

$$n^{-1} r_{[sn]} = n^{-1} \sum_t \mathbf{1}\left(Z_t < Z_{[sn]}\right) = n^{-1} \sum_t \mathbf{1}\left(a_n^{-1} Z_t < a_n^{-1} Z_{[sn]}\right)$$

$$= \sum_{t=1}^{n} \int_{\frac{t-1}{n}}^{\frac{t}{n}} \mathbf{1}\left(a_n^{-1} Z_{[\frac{t}{n}n]} < a_n^{-1} Z_{[sn]}\right) du = \int_0^1 \mathbf{1}\left(a_n^{-1} Z_{[\frac{t}{n}n]} < a_n^{-1} Z_{[sn]}\right) du$$

$$\Rightarrow R(s) \equiv \int_0^1 \mathbf{1}\left(L_\alpha(u) < L_\alpha(s)\right) du \quad \text{as } n \to \infty\,. \tag{6.5.3}$$

Therefore, the limit of normalized partial sums of ranks defines a stochastic process indexed by $s \in [0,\ 1]$ such that $R(s)$ is the occupation time of the set $(-\infty,\ L_\alpha(s)]$ by the *Levy* motion.

Let $R(0) = \int_0^1 \mathbf{1}\left(L_\alpha(u) < 0\right) du$, follow the some line of Property 2 of Breitung and Gourieroux (1997), we can show that $R(s) = sR_1(0) + (1-s)R_2(0)$, where $R_1(0)$ and $R_2(0)$ are independent.

Furthermore, by noticing that $r_{[sn]} = r_{t-1}$, for $\frac{t-1}{n} \le s < \frac{t}{n}$, $t = 1, \ldots, n$, we have the following lemma:

Lemma 6.7 *Let Z_t be generated by (6.5.1) and r_t be the ranks of Z_t defined in (6.5.2), as $n \to \infty$, we have*

(a) $n^{-2}\sum_{t=1}^{n} r_{t-1}(r_t - r_{t-1}) \Rightarrow \int_0^1 R(s)dR(s)$,

(b) $n^{-3}\sum_{t=1}^{n} r_{t-1}^2 \Rightarrow \int_0^1 R(s)^2 ds$,

(c) $n^{-2}\sum_{t=1}^{n} (r_t - r_{t-1})^2 \Rightarrow \int_0^1 (dR(s))^2$.

Proof. Part (a) is proved by noticing that

$$n^{-2}\sum_{t=1}^{n} r_{t-1}(r_t - r_{t-1}) = n^{-2}\sum_{t=1}^{n} \int_{\frac{t-1}{n}}^{\frac{t}{n}} r_{[sn]} dr_{[sn]} = \int_0^1 \left(n^{-1} r_{[sn]}\right) d\left(n^{-1} r_{[sn]}\right) \Rightarrow \int_0^1 R(s)dR(s)$$

For part (b), it is clear that

$$n^{-3}\sum_{t=1}^{n} r_{t-1}^2 = n^{-2}\sum_{t=1}^{n} \int_{\frac{t-1}{n}}^{\frac{t}{n}} r_{[sn]}^2 ds = \int_0^1 \left(n^{-1} r_{[sn]}\right)^2 ds \Rightarrow \int_0^1 R(s)^2 ds,$$

and finally, for part (c)

$$n^{-2}\sum_{t=1}^{n} (r_t - r_{t-1})^2 = n^{-2}\sum_{t=1}^{n} \int_{\frac{t-1}{n}}^{\frac{t}{n}} \left(dr_{[sn]}\right)^2 = \int_0^1 \left(dn^{-1} r_{[sn]}\right)^2 \Rightarrow \int_0^1 (dR(s))^2,$$

where $\int_0^1 (dR(s))^2$ is the quadratic variation of $R(s)$ defined as

$$\int_0^1 (dR(s))^2 \equiv [R, R]_1 = R(1)^2 - 2\int_0^1 R(s)dR(s).$$

Thus we complete the proof of Lemma 6.7.

Combining (a) and (b), we have

$$n\left(\sum_{t=1}^{n} r_{t-1}(r_t - r_{t-1}) \Big/ \sum_{t=1}^{n} r_{t-1}^2\right) \Rightarrow \int_0^1 R(s)dR(s) \Big/ \int_0^1 R(s)^2 ds, \tag{6.5.4}$$

hence

$$\sum_{t=1}^{n} r_{t-1}(r_t - r_{t-1}) \Big/ \sum_{t=1}^{n} r_{t-1}^2 = o_p(1). \tag{6.5.5}$$

Define $c_j = \sum_{t=0}^{n-1} r_{t+j} r_t$, $j = 0, 1$, and $\sigma_*^2 = (n-1)^{-1} \sum_{t=1}^{n} \left[r_t - (c_1/c_0) r_{t-1}\right]^2$, then by (6.5.5)

$$c_1/c_0 - 1 = \sum_{t=1}^{n} r_{t-1}(r_t - r_{t-1}) \Big/ \sum_{t=1}^{n} r_{t-1}^2 = o_p(1), \tag{6.5.6}$$

and

$$\begin{aligned} n^{-1}\sigma_*^2 &= n^{-1}(n-1)^{-1} \sum_{t=1}^{n} \left[r_t - (c_1/c_0) r_{t-1}\right]^2 \approx n^{-2} \sum_{t=1}^{n} \left[r_t - r_{t-1} - (c_1/c_0 - 1) r_{t-1}\right]^2 \\ &= n^{-2} \sum_{t=1}^{n} (r_t - r_{t-1})^2 - 2n^{-2}(c_1/c_0 - 1) \sum_{t=1}^{n} r_{t-1}(r_t - r_{t-1}) + n^{-2}(c_1/c_0 - 1)^2 \sum_{t=1}^{n} r_{t-1}^2 \\ &\Rightarrow \int_0^1 (dR(s))^2 . \end{aligned} \tag{6.5.7}$$

The rank counterpart of the conventional *DF t* statistic, suggested by Breitung and Gourieroux (1997), is defined as

$$t_\rho = \hat{\sigma}_*^{-1} (c_1 - c_0)/c_0^{1/2} .$$

Collecting result (6.5.6) and (6.5.7), we can establish that

$$t_\rho = \sigma_*^{-1}(c_1 - c_0)/c_0^{1/2} = \frac{n^{-2} \sum_{t=1}^{n} r_t (r_t - r_{t-1})}{\left\{\left(n^{-1}\sigma_*^2\right)\left(n^{-3} \sum_{t=1}^{n} r_{t-1}^2\right)\right\}^{1/2}} \Rightarrow \int_0^1 R(s)dR(s) \Big/ \left\{ \int_0^1 (dR(s))^2 \int_0^1 R(s)^2 ds \right\}^{1/2},$$

thus, the following theorem is in order:

Theorem 6.6 *Under model (6.5.1), the rank t statistic for testing unit root has the following asymptotic distribution*

$$t_\rho \Rightarrow \int_0^1 R(s)dR(s) \Big/ \left\{ \int_0^1 (dR(s))^2 \int_0^1 R(s)^2 ds \right\}^{1/2} \quad \text{as } n \to \infty .$$

From this theorem, we see that the asymptotic distribution of t-ratio for the ranks is a functional of a stochastic process $R(s)$, which is the occupation time of the set $\left(-\infty, L_\alpha(s)\right]$ by a standard $S\alpha S$ *Levy* motion $L_\alpha(s)$.

6.6 Asymptotic Behaviors of Spurious Regression for Infinite Variance Case

The 'nonsense' of regression between two random walks was empirically evident in Granger and Newbold (1974). Phillips (1986) obtained some analytical results for the spurious regression for the finite variance case. The purpose of section is to study the spurious regression when the error variance is infinite. We will show that the 'nonsense' results are also valid if regression is made between two independent random walks whose errors are from the domain of attraction of symmetric stable laws.

Consider the following regression

$$Y_t = \beta_0 + \beta_1 X_t + u_t, \; t = 1, \; \ldots, \; n . \tag{6.6.1}$$

If $\{Y_t\}$ and $\{X_t\}$ are generated by two independent random walks

$$Y_t = Y_{t-1} + v_t, \; X_t = X_{t-1} + w_t , \tag{6.6.2}$$

then we encounter the so-called spurious regression. Phillips (1986) studied the asymptotic behaviors of sample moments for spurious regression in the case that $\{v_t\}$ and $\{w_t\}$ are sequences satisfying some weak dependencies and having finite variances. In this section, we assume $\{v_t\}$ and $\{w_t\}$ to be two independent sequences of *iid* random variables in the domains of attraction of a $S_\alpha(\sigma_v,0,0)$ law and a $S_\alpha(\sigma_w,0,0)$ law with $0<\alpha<2$ respectively. Assuming $Y_0 = X_0 = 0$ for simplicity, the following lemma is then in order

Lemma 6.8 *Let* $\{Y_t\}$ and $\{X_t\}$ *be generated by (6.6.2). If the innovation sequences* $\{v_t\}$ *and* $\{w_t\}$ *be two independent sequences of iid random variables in the domains of attraction of a* $S_\alpha(\sigma_v,0,0)$ *law and a* $S_\alpha(\sigma_w,0,0)$ *law with* $0<\alpha<2$ *respectively, then, as* $n\to\infty$,

(a) $$(na_n)^{-1}\sum_{t=1}^{n} X_t \Rightarrow C_\alpha^{-1\,\alpha}\sigma_w \int_0^1 W_\alpha(r)dr,$$

$$(na_n)^{-1}\sum_{t=1}^{n} Y_t \Rightarrow C_\alpha^{-1\,\alpha}\sigma_v \int_0^1 V_\alpha(r)dr;$$

(b) $$\left(na_n^2\right)^{-1}\sum_{t=1}^{n} X_t^2 \Rightarrow \left(C_\alpha^{-1\,\alpha}\sigma_w\right)^2 \int_0^1 W_\alpha(r)^2 dr,$$

$$\left(na_n^2\right)^{-1}\sum_{t=1}^{n} Y_t^2 \Rightarrow \left(C_\alpha^{-1\,\alpha}\sigma_v\right)^2 \int_0^1 V_\alpha(r)^2 dr;$$

(c) $$\left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(X_t-\bar{X}\right)^2 \Rightarrow \left(C_\alpha^{-1\,\alpha}\sigma_w\right)^2\left\{\int_0^1 W_\alpha(r)^2 dr - \left[\int_0^1 W_\alpha(r)dr\right]^2\right\},$$

$$\left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(Y_t-\bar{Y}\right)^2 \Rightarrow \left(C_\alpha^{-1\,\alpha}\sigma_v\right)^2\left\{\int_0^1 V_\alpha(r)^2 dr - \left[\int_0^1 V_\alpha(r)dr\right]^2\right\};$$

(d) $$\left(na_n^2\right)^{-1}\sum_{t=1}^{n} Y_t X_t \Rightarrow \left(C_\alpha^{-1\,\alpha}\right)^2 \sigma_v\sigma_w \int_0^1 V_\alpha(r)W_\alpha(r)dr;$$

(e) $\left(a_n^2\right)^{-1}\sum_{t=1}^{n} X_t(X_t - X_{t-1}) \Rightarrow (1/2)\sigma_w^2\left\{C_\alpha^{-2/\alpha}W_\alpha(1)^2 + C_{\alpha/2}^{-2/\alpha}L_w\right\}$,

$$\left(a_n^2\right)^{-1}\sum_{t=1}^{n} Y_t(Y_t - Y_{t-1}) \Rightarrow (1/2)\sigma_v^2\left\{C_\alpha^{-2/\alpha}V_\alpha(1)^2 + C_{\alpha/2}^{-2/\alpha}L_v\right\};$$

(f) $\left(a_n^2\right)^{-1}\sum_{t=1}^{n} Y_{t-1}(X_t - X_{t-1}) + \left(a_n^2\right)^{-1}\sum_{t=1}^{n} X_{t-1}(Y_t - Y_{t-1})$

$$\Rightarrow C_\alpha^{-2/\alpha}\sigma_v\sigma_w\left\{V_\alpha(1)W_\alpha(1) - \int_0^1 dW_\alpha(r)dV_\alpha(r)\right\};$$

where $W_\alpha(r)$ and $V_\alpha(r)$ *are independent standard SαS Levy processes on* $D[0,1]$, $L_v \sim S_{\alpha/2}(1,1,0)$, $L_w \sim S_{\alpha/2}(1,1,0)$, *and* L_v and L_w *are independent.*

Proof. Result (a) and (b) can be found in Theorem 6.1. Observing that

$$\left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2 = \left(na_n^2\right)^{-1}\sum_{t=1}^{n} X_t^2 - \left(a_n^{-1}\sum_{t=1}^{n} X_t\right)^2.$$

Part (c) follows immediately. To prove part (d), we first note that

$$\begin{aligned} a_n^{-2}\sum_{t=0}^{n} v_t X_{t-1} &= \sum_{t=1}^{n} a_n^{-1} X_{t-1}\int_{\frac{t-1}{n}}^{\frac{t}{n}} dY_n(r) = \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}} X_n(r)dY_n(r) \\ &= \int_0^1 X_n(r)dY_n(r) \Rightarrow C_\alpha^{-2/\alpha}\sigma_v\sigma_w\int_0^1 W_\alpha(r)dV_\alpha(r). \end{aligned} \qquad (6.6.3)$$

and

$$a_n^{-2}\sum_{t=0}^{n} w_t Y_{t-1} \Rightarrow C_\alpha^{-2/\alpha}\sigma_v\sigma_w\int_0^1 V_\alpha(r)dW_\alpha(r). \qquad (6.6.4)$$

By Lemma 6.6,

$$a_n^{-2}\sum_{t=0}^{n} v_t w_t \to 0, \quad a.s.. \qquad (6.6.5)$$

Combining (6.6.3), (6.6.4) and (6.6.5), we obtain that

$$\left(na_n^2\right)^{-1}\sum_{t=1}^{n} Y_t X_t = \left(na_n^2\right)^{-1}\sum_{t=1}^{n} Y_{t-1}X_{t-1} + \left(na_n^2\right)^{-1}\left\{\sum_{t=1}^{n} v_t X_{t-1} + \sum_{t=1}^{n} w_t Y_{t-1} + \sum_{t=1}^{n} v_t w_t\right\}$$
$$= \sum_{t=1}^{n} \int_{\frac{t-1}{n}}^{\frac{t}{n}} Y_n(r)X_n(r)dr + o_p(1) = \int_0^1 Y_n(r)X_n(r)dr + o_p(1)$$
$$\Rightarrow \left(C_a^{-1\,a}\right)^2 \sigma_v \sigma_w \int_0^1 V_a(r)W_a(r)dr.$$

This proves part (d). To resolve (e), we proceed as follows

$$\left(a_n^2\right)^{-1}\sum_{t=1}^{n} X_t(X_t - X_{t-1}) = \left(a_n^2\right)^{-1}\sum_{t=1}^{n}(X_{t-1} + w_t)w_t$$
$$= \left(a_n^2\right)^{-1}\sum_{t=1}^{n} w_t X_{t-1} + \left(a_n^2\right)^{-1}\sum_{t=1}^{n} w_t^2$$
$$\Rightarrow (1/2)\sigma_w^2\left\{C_a^{-2\,a}W_a(1)^2 - C_{a\,2}^{-2\,a}L_w\right\} + \sigma_w^2 C_{a\,2}^{-2\,a}L_w$$
$$= (1/2)\sigma_w^2\left\{C_a^{-2\,a}W_a(1)^2 + C_{a\,2}^{-2\,a}L_w\right\}$$

For part (f), using that $\sum_{t=1}^{n} v_t \sum_{t=1}^{n} w_t = \sum_{t=1}^{n} v_t w_t + \sum_{t=1}^{n}\left(\sum_{j=1}^{t-1} v_j\right)w_t + \sum_{t=1}^{n}\left(\sum_{j=1}^{t-1} w_j\right)v_t$ and the fact that

$\left(a_n^2\right)^{-1}\sum_{t=1}^{n} v_t w_t \to 0, a.s.$, we have

$$\left(a_n^2\right)^{-1}\sum_{t=1}^{n} Y_t(X_t - X_{t-1}) + \left(a_n^2\right)^{-1}\sum_{t=1}^{n} X_t(Y_t - Y_{t-1})$$
$$= \left(a_n^2\right)^{-1}\sum_{t=1}^{n} w_t Y_{t-1} + \left(a_n^2\right)^{-1}\sum_{t=1}^{n} v_t X_{t-1} + 2\left(a_n^2\right)^{-1}\sum_{t=1}^{n} v_t w_t$$
$$= \left(a_n^2\right)^{-1}\sum_{t=1}^{n} w_t \sum_{t=0}^{n} v_t + \left(a_n^2\right)^{-1}\sum_{t=1}^{n} v_t w_t$$
$$\Rightarrow C_a^{-2\,a}\sigma_v\sigma_w V_a(1)W_a(1).$$

We complete the proof of Lemma 6.8.

Theorem 6.8 *Suppose (6.6.1) is estimated by least square regression and the conditions of Lemma 6.8 are satisfied. Then, as* $n \to \infty$,

(a) $$\hat{\beta}_1 \Rightarrow \frac{\sigma_v \left\{ \int_0^1 V_a(r) W_a(r) dr - \int_0^1 V_a(r) dr \int_0^1 W_a(r) dr \right\}}{\sigma_w \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\}} = (\sigma_v / \sigma_w) \zeta \text{, where}$$

$$\zeta = \frac{\left\{ \int_0^1 V_a(r) W_a(r) dr - \int_0^1 V_a(r) dr \int_0^1 W_a(r) dr \right\}}{\left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\}};$$

(b) $$a_n^{-1} \hat{\beta}_0 \Rightarrow C_a^{-1\,a} \sigma_v \left\{ \int_0^1 V_a(r) dr - \zeta \int_0^1 W_a(r) dr \right\};$$

(c) $$n^{-1\,2} t_{\beta_1} \Rightarrow \mu / v^{1\,2}, \text{ where}$$

$$\mu = \int_0^1 V_a(r) W_a(r) dr - \int_0^1 V_a(r) dr \int_0^1 W_a(r) dr,$$

$$v = \left\{ \int_0^1 V_a(r)^2 dr - \left(\int_0^1 V_a(r) dr \right)^2 \right\} \times \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\}$$

$$- \left\{ \int_0^1 V_a(r) W_a(r) dr - \int_0^1 V_a(r) dr \int_0^1 W_a(r) dr \right\}^2;$$

(d) $$n^{-1\,2} t_{\beta_0} \Rightarrow \left\{ \int_0^1 V_a(r) dr - \zeta \int_0^1 W_a(r) dr \right\}$$

$$\times \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\} \Big/ \left[v \int_0^1 W_a(r)^2 dr \right]^{1\,2};$$

(e) $$R^2 \Rightarrow \frac{\zeta^2 \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\}}{\int_0^1 V_a(r)^2 dr - \left(\int_0^1 V_a(r) dr \right)^2};$$

(f) $DW \to 0$ $a.s.$, and

$$nDW \Rightarrow \left\{ \left(L_v / C_a^{-2\,a} \sigma_v^2 \right) + \zeta^2 \left(L_w / C_a^{-2\,a} \sigma_w^2 \right) \right\} \left[\int_0^1 V_a(r)^2 dr - \left(\int_0^1 V_a(r) dr \right)^2 \right.$$

$$\left. - \zeta^2 \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\} \right].$$

Proof. Note that $\hat{\beta}_1 = \dfrac{\sum_{t=1}^{n} Y_t\left(X_t - \bar{X}\right)}{\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2} = \dfrac{\left(na_n^2\right)^{-1}\sum_{t=1}^{n} Y_t X_t - \left(a_n^{-1}\sum_{t=1}^{n} Y_t\right)\left(a_n^{-1}\sum_{t=1}^{n} X_t\right)}{\left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2}$, and

$a_n^{-1}\hat{\beta}_0 = a_n^{-1}\left(\bar{Y} - \hat{\beta}_1\bar{X}\right) = \left(na_n\right)^{-1}\sum_{t=1}^{n} Y_t - \hat{\beta}_1\left(na_n\right)^{-1}\sum_{t=1}^{n} X_t$, applying Theorem 6.1, part (a) and (b) follow immediately.

To prove part (c), we define $s^2 = n^{-1}\sum_{t=1}^{n}\left(Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_t\right)^2$, then

$$a_n^{-2}s^2 = \left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left\{Y_t - \bar{Y} - \hat{\beta}_1\left(X_t - \bar{X}\right)\right\}^2 = \left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(Y_t - \bar{Y}\right)^2 - \left(na_n^2\right)^{-1}\hat{\beta}^2\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2$$

$$\Rightarrow C_\alpha^{-2/\alpha}\sigma_v^2\left[\int_0^1 V_\alpha(r)^2 dr - \left(\int_0^1 V_\alpha(r)dr\right)^2 - \zeta^2\left\{\int_0^1 W_\alpha(r)^2 dr - \left(\int_0^1 W_\alpha(r)dr\right)^2\right\}\right].$$

Now notice that $n^{-1/2}t_{\hat{\beta}_1} = \dfrac{\hat{\beta}_1}{n^{1/2}s_{\hat{\beta}_1}} = \dfrac{\hat{\beta}_1}{n^{1/2}s\left(\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2\right)^{-1/2}} = \dfrac{\hat{\beta}_1}{a_n^{-1}s\left(\left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2\right)^{-1/2}}$,

Part (c) is proved by using Lemma 6.7 and results in Theorem 6.5 (a) and (b) after some simple arithmetic

$$n^{-1/2}t_{\hat{\beta}_1} \Rightarrow \mu/\nu^{1/2},$$

where μ and ν are defined in part (c) of Theorem 6.8.

Part (d) can be established similar fashion by observing that

$$n^{-1/2}t_{\hat{\beta}_0} = \frac{\hat{\beta}_0}{n^{1/2}s_{\hat{\beta}_0}} = \frac{\hat{\beta}_0\left(n\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2\right)^{1/2}}{n^{1/2}s\left(\sum_{t=1}^{n}X_t^2\right)^{1/2}} = \frac{\left(a_n^{-1}\hat{\beta}_0\right)\left\{\left(na_n^2\right)^{-1}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2\right\}^{1/2}}{\left(a_n^{-1}s\right)\left\{\left(na_n^2\right)^{-1}\sum_{t=1}^{n}X_t^2\right\}^{1/2}}$$

The coefficient of determination converges as follows

$$R^2 = \frac{SSR}{SSTO} = \frac{\sum_{t=1}^n (\hat{Y}_t - \bar{Y})^2}{\sum_{t=1}^n (Y_t - \bar{Y})^2} = \frac{\hat{\beta}_1^2 (na_n^2)^{-1} \sum_{t=1}^n (X_t - \bar{X})^2}{(na_n^2)^{-1} \sum_{t=1}^n (Y_t - \bar{Y})^2}$$

$$\Rightarrow \frac{(\sigma_v \ \sigma_w)^2 \varsigma^2 (C_a^{-1\,a} \sigma_w)^2 \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\}}{(C_a^{-1\,a} \sigma_v)^2 \left\{ \int_0^1 V_a(r)^2 dr - \left(\int_0^1 V_a(r) dr \right)^2 \right\}}$$

$$= \frac{\varsigma^2 \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\}}{\int_0^1 V_a(r)^2 dr - \left(\int_0^1 V_a(r) dr \right)^2}.$$

The Durbin-Watson statistic is given by

$$DW = \frac{\sum_{t=2}^n (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^n \hat{u}_t^2} = n^{-1} \frac{a_n^{-2} \sum_{t=2}^n (v_t - \hat{\beta}_1 w_t)^2}{(na_n^2)^{-1} \sum_{t=2}^n (Y_t - \bar{Y} - \hat{\beta}_1 (X_t - \bar{X}))^2}.$$

The denominator converges as

$$a_n^{-2} \sum_{t=2}^n (v_t - \hat{\beta}_1 w_t)^2 \Rightarrow L_v + C_a^{-2\,a} \sigma_v^2 \varsigma^2 L_w / \sigma_w^2 \quad \text{as } n \to \infty,$$

whereas the numerator converges as

$$(na_n^2)^{-1} \sum_{t=2}^n (Y_t - \bar{Y} - \hat{\beta}_1 (X_t - \bar{X}))^2 \Rightarrow C_a^{-2\,a} \sigma_a^2 \left[\int_0^1 V_a(r)^2 dr - \left(\int_0^1 V_a(r) dr \right)^2 \right.$$
$$\left. - \varsigma^2 \left\{ \int_0^1 W_a(r)^2 dr - \left(\int_0^1 W_a(r) dr \right)^2 \right\} \right].$$

Thus, $DW \xrightarrow{p} 0$. However the standardized DW statistic converges as

$$nDW \Rightarrow \left(C_\alpha^{2/\alpha}\sigma_v^{-2}L_v + \zeta^2\sigma_w^{-2}L_w\right)\Big/\Big[\int_0^1 V_\alpha(r)^2dr - \left(\int_0^1 V_\alpha(r)dr\right)^2$$

$$-\zeta^2\left\{\int_0^1 W_\alpha(r)^2dr - \left(\int_0^1 W_\alpha(r)dr\right)^2\right\}.$$

This completes the proof of Theorem 6.8.

Remark. For the infinite-variance case, we also have the phenomenon of a spurious regression in the sense of Granger and Newbold (1974). In other words, the least squares regression in (6.6.1) leads to the divergence of the *OLS* estimate of $\hat{\beta}_0$, and to the convergence of $\hat{\beta}_1$. The coefficient of determinant R^2 converges to a random variable, conventional *t*-ratios diverge with rate $n^{1/2}$ and the *DW*-statistic is $O_p\left(n^{-1}\right)$.

6.7 Concluding Remarks

This chapter considers the asymptotic properties of sample moments and some unit root test statistics for the first-order autoregressive time series models with infinite variances. The results obtained in this chapter can be viewed as a parallel but not trivial extension of the finite-variance case. Some asymptotic distributions of sample moments are found to have explicit densities. The limiting distributions for the *LM* statistic and the *DW* statistics are expressed as functionals of standard *SαS Levy* motions. The ranked Dickey-Fuller test converges to a functional of some stochastic process other than Levy motion. The spurious phenomenon for the infinite-variance case is observed to have the similar fashion as the Gaussian case. Some additional remarks are made as follows: (i). We assume that the innovations are symmetric throughout this chapter. But this symmetry condition may be relaxed. When $\alpha < 1$ no further requirement beyond the domain of attraction of an a-stable law seems to be needed. When $\alpha > 1$ we require $E(\varepsilon_t) = 0$ so that the sums involving ε_t do not need to be centered. Only for the case of $\alpha = 1$, we assume the symmetry. In fact, Chan and Tran (1989) derived the asymptotic results based on the above assumptions. (ii). Similar to the finite variance case, all the results can be extended easily to models with drifts and time trends by just replacing the integrals of *Levy* process by demeaned or detrended *Levy* processes. (iii). If $\varepsilon = A^{1/2}(G_1, \ldots, G_n)$ where G_t's are *iid* normal, then the scale invariant statistics, such as the *LM* statistics, the *DW* statistics, have the same asymptotic distributions as it is for the normal case. Note that, in this case, ε_t's are identically distributed but not stochastically independent. If A is a positive $\alpha/2$ stable random variable, then ε_t's are jointly $S\alpha S$, and hence have infinite variance, but they are not stochastically independent.

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IMAGE EVALUATION TEST TARGET (QA-3)

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